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EFFICIENT ALGORITHM OF THE BUCKLING LENGTHS METHOD IN ANALYSIS OF SPATIAL FRAME SYSTEMS FOR STABILITY

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Abstract: An effective computational algorithm has been developed that allows finding the buckling lengths of the rod elements of frame structures. A matrix of unit reaction from the side the rest of the structure is formed for each element and eigenvalues problem of longitudinal bending equation of the rod is solved. When searching for eigenvalues, the stress state in all elements of the system except the tested rod is fixed. The algorithm takes into account the interaction of each studied element with all other elements of the structure. The constructed algorithm allows, in a limited time, commensurate with the expenditure of a conventional static calculation, to find an exact solution to the problem of eigenvalues for each rod and to determine the buckling lengths of all rod elements included in the structure.

Keywords: buckling, buckling of the rod systems, buckling length, effective length, check local buckling

ЭФФЕКТИВНЫЙ АЛГОРИТМ МЕТОДА РАСЧЕТНЫХ ДЛИН ПРИ АНАЛИЗЕ УСТОЙЧИВОСТИ ПРОСТРАНСТВЕННЫХ РАМНЫХ СИСТЕМ

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Аннотация: Построен эффективный вычислительный алгоритм, позволяющий находить расчетные длины стержневых элементов рамных конструкций. Для каждого элемента формируется матрица единичных реакций со стороны остальной конструкции и, с учетом этих реакций, решается задача о собственных значениях уравнения продольного изгиба стержня. При поиске собственных значений напряженное состояние во всех элементах системы, кроме тестируемого стержня, фиксируется. Алгоритм учитывает взаимодействие каждого исследуемого элемента со всеми остальными элементами конструкции. Построенный алгоритм позволяет за ограниченное время, соизмеримое с затратами на обычный статический расчет, найти точное решение задачи о собственных значениях для каждого стержня и определить расчетные длины всех стержневых элементов входящих в конструкцию.

Ключевые слова: устойчивость, устойчивость стержневых систем, расчетная длина, свободная длина, проверка локальной устойчивости

1. INTRODUCTION

Checking the local stability of rod elements is one of the most important checks necessary to ensure reliable operation of building structures. A significant contribution to the development of methods for calculating the local stability of rod elements was made by both foreign and domestic authors. Here, first of all, it is necessary to note the works of G.I. Belyj [1], A.V. Alexandrov [2,3], I.D. Grudev [4], G.A. Manuilov1 [5], A.M. Belostotsky [6,7], A.V. Perelmuter, V.I. Slivker [8] and many other authors. In foreign publications, the modern concept of the method for checking the local stability of rod elements, based on the concept of the calculated length, is presented in the work of Y. Duan and W.F. Chen [9].Currently, two methods have been established for checking the local stability of frame structures - the direct analysis method [10,11,12] (a number of standards allow the direct setting of the initial irregularities of the rod elements within the framework of the P-Delta analysis [13,14]) and a method based on the specifying of the effective lengths of rod elements. The value of the effective length method is that it provides a simple approach to assessing the stability of frame members. It is based on solving the problem of longitudinal bending of a hinged rod with a sinusoidal shape of the initial roughness. As a result, the so-called buckling factor φ for central compression is found, which makes it possible to calculate the longitudinal force at which a stress equal to the design resistance R_y arises in one of the fibers of the rod (in European standards this coefficient is called "reduction" factor for the relevant buckling mode" and is indicated by the letter χ). The canonical form defining the coefficient φ has the form

$$\varphi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}}, \tag{1}$$

where $\bar{\lambda} = \frac{\lambda}{\pi}\sqrt{\frac{R_y}{E}}$ is the reduced slenderness of the rod, $\lambda = \frac{l}{i}$ is the slenderness of the rod, E is the elastic modulus of the material, and the parameter ϕ is calculated by the formula: $\phi = \frac{1}{2}(1 + \bar{e} + \bar{\lambda}^2)$, where $\bar{e} = \frac{Af_0}{W}$ is the reduced eccentricity of the roughness normalized in a certain way (here A and W are the area and modulus of section). Domestic standards СП 16.13330.2017 [15] use the same formula with some ritual changes. When moving to the case of arbitrary restrictions of the ends of the rod, it is proposed to substitute in the formula for calculating the slenderness λ, instead of the physical length of the rod l, the effective length of the rod l_p – the distance between the inflection points on the buckling shape curve. This substitution is justified by the fact that in this way an arbitrary rod with arbitrary boundary conditions is, as it were, reduced to a hingedly supported one. In [16] it was shown that, indeed, if we take the form of the initial irregularity proportional to the form of buckling of the rod under the appropriate boundary conditions, then formula (1) remains valid, and the role of the physical length of the rod is transferred to the effective length. This proof can be considered some justification for the method of checking the local stability of rods based on the concept of the "effective length" of the rod, although the approach itself requires additional research.

But, whatever it was, the key problem in the methodology for checking the local stability of rods is the determination of these same effective lengths. Many works are devoted to this problem. Since in the general case it is not possible to determine the boundary conditions for each individual rod (the boundary conditions for an individual rod are determined by the system of pushback reactions from all other elements of the structure), a certain fragment of the structure adjacent to the tested rod is identified and, introducing some simplifying assumptions, solve for the selected rod the problem of bucklnug [17-28]. Based on the found eigenvalues, the required effective length is determined. In this regard, it is worth mentioning the important work of R.H. Wood [17], which established the theoretical basis for the calculation of buckling in the European BS EN 1993 [13], as well as in the American standards AISC/AISC 360-16 [14]. For a more accurate analysis of buckling, F. Cheong-Siat-Moi [18] drew attention to the need to take into account the influence on the behavior of an individual member, both the elements surrounding it, and the influence of the entire system. J. Heldesland and R. Bjørhovde [20] showed the importance of fully accounting for the contribution of adjacent elements to the rotational stiffness of the ends of the element. R.K. Bridge and D.J. Fraser drew attention to the fact that if a significant longitudinal force acts in the elements adjacent the member, this can weaken its rigidity and reduce the critical force [19]. To solve the problem of lightly loaded members, A.V. Perelmuter and V.I. Slivker [8] applied the method of "additional loading of passive elements". The "additional loading" method was implemented in the industrial software package SCAD [33]. Many other authors contributed to the solution of the problem of local stability of rods based on the effective length method [21-29]. And this list of works is far from exhaustive. A good overview of works in this direction can be found, for example, in [29]. The disadvantage of this approach is that, firstly, the selected fragment of the structure does not always reliably provide real boundary conditions for the element under study. Secondly, the critical load of an individual rod inside a frame structure cannot be obtained without taking into account the forces in the remaining elements of the structure, since the stress state in the remaining elements of the structure affects the stiffness of the end reactions of the selected element and changes its effective length. And thirdly, since each element requires an individual approach when choosing a calculation fragment, this approach does not allow us to formalize the problem of determining effective lengths. This complicates its implementation within computer systems. Thus, following this path, the engineer is forced to either make difficult arithmetic calculations in advance, or use various reference books [13-15,30], which, as a rule, makes the analysis even more inaccurate.

In an effort to avoid complex mathematical calculations, engineers very often use the results of a structural buckling analysis to assign effective lengths to structural members. The calculation for overall stability involves a proportional increase in the forces in the elements to the level at which the structure loses stability, and the resulting values of the critical forces in the elements are used to determine the effective lengths. This approach could be considered valid if the actual internal forces were equal to the critical ones. In fact, the actual distribution of forces is always much less than the critical one. An overestimation of the axial forces leads to a softening of the boundary conditions for each individual rod. As a result, with this approach, the effective length of all rod elements always turns out to be overestimated.

This paper presents a universal algorithm that allows you to determine the effective lengths of all rod elements that make up a structure, regardless of its complexity and taking into account the real stress state in the structural elements, allowing you to fully automate this procedure.

2. ANALYSIS

Consider the structure depicted in Figure 1a. Let us want to find the effective length of the rod attached to nodes r and s. To do this, it is necessary to construct a matrix of unit reactions of the system reduced to nodes r and s (Fig. 1b). Let's denote this matrix $\boldsymbol{S_{rs}}$. Next, we exclude from this matrix the unit reactions of the element $r - s$ itself, and instead of them we add the reactions of the same element as a function of the critical parameter $\alpha = l\sqrt{N/EI}$, where l – rod length, E – modulus of elasticity of the material, I is the moment of inertia of the cross section of the rod, and N– the value of the axial force in the element. Let us denote the own matrix of unit reactions of the tested rod $\boldsymbol{S_M}(\alpha)$.

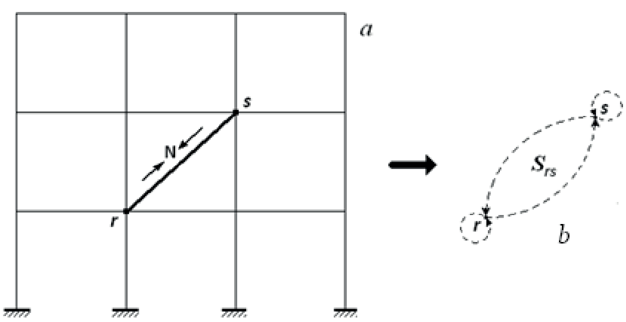

Figure 1. a – design model of the structure; b – design model reduced to the attachment points of the tested element – all superimposed restrictions of the main system, except those superimposed on nodes r and s, are reset (dotted lines with arrows conventionally depict the mutual influence of the nodes)

Thus, the task of determining the effective length of the element $r - s$ reduces to solving the equation

$$det\ [\boldsymbol{S}_{rs} - \boldsymbol{S}_M(\alpha_0) + \boldsymbol{S}_M(\alpha)] = 0 \qquad (2)$$

relative to the parameter α (here $\boldsymbol{S}_M(\alpha_0)$ – the matrix of unit reactions of the element $r - s$ at the current value of the longitudinal force N_0). For example, for a rigidly clamped rod (plane problem) in the local coordinate system of the rod, the matrix $\boldsymbol{S}_M(\alpha)$ has the form [31]

$$\boldsymbol{S}_M(\alpha) = \begin{pmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3}\eta_1 & \frac{6EI}{l^2}\eta_2 & 0 & -\frac{12EI}{l^3}\eta_1 & \frac{6EI}{l^2}\eta_2 \\ 0 & \frac{6EI}{l^2}\eta_2 & \frac{4EI}{l}\eta_3 & 0 & -\frac{6EI}{l^2}\eta_2 & \frac{2EI}{l}\eta_4 \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12EI}{l^3}\eta_1 & -\frac{6EI}{l^2}\eta_2 & 0 & \frac{12EI}{l^3}\eta_1 & -\frac{6EI}{l^2}\eta_2 \\ 0 & \frac{6EI}{l^2}\eta_2 & \frac{2EI}{l}\eta_4 & 0 & -\frac{6EI}{l^2}\eta_2 & \frac{4EI}{l}\eta_3 \end{pmatrix} \qquad (3)$$

where

$$\eta_1 = \frac{\left(\frac{\alpha}{2}\right)^3}{3\left(\operatorname{tg}\frac{\alpha}{2} - \frac{\alpha}{2}\right)}, \quad \eta_2 = \frac{\left(\frac{\alpha}{2}\right)^2 \operatorname{tg}\frac{\alpha}{2}}{3\left(\operatorname{tg}\frac{\alpha}{2} - \frac{\alpha}{2}\right)},$$
$$\eta_3 = \frac{1 - \frac{\alpha}{\operatorname{tg}\alpha}}{\frac{8}{\alpha}\left(\operatorname{tg}\frac{\alpha}{2} - \frac{\alpha}{2}\right)}, \quad \eta_4 = \frac{\frac{\alpha}{\sin\alpha} - 1}{\frac{4}{\alpha}\left(\operatorname{tg}\frac{\alpha}{2} - \frac{\alpha}{2}\right)}. \qquad (4)$$

And, of course, before including the matrices $\boldsymbol{S}_M(\alpha_0)$ and $\boldsymbol{S}_M(\alpha)$ in equation (2), they must be converted to the global coordinate system. After the solution to equation (2) has been found, the value of the effective length coefficient is calculated according to the formula $\mu = \pi/\alpha$ (effective length coefficient – the ratio of the effective length of the rod to the physical length). The main difficulty of the described procedure is the construction of a stiffness matrix reduced to the attachment points of the tested rod (matrix $\boldsymbol{S}_{rs}$) . If, for example, we use the Gaussian elimination procedure for this, we will need to place the rows corresponding to the nodes of the tested rod at the end of the system stiffness matrix and process them with all the overlying rows. That is, in fact, it is necessary to perform a direct sweep along the entire matrix of the system. And this procedure will need to be performed for each rod. It is clear that if our structure consists of hundreds or thousands of rods, such a procedure becomes impossible.

3. FORMATION OF THE SYSTEM STIFFNESS MATRIX REDUCED TO NODES r AND s.

Let's assume that we somehow managed to construct a matrix inverse to the original stiffness matrix of the system – the flexibility matrix. Let's denote this matrix by the letter ***F***. Note that the physical meaning of the coefficients in this matrix f_{ij} is movement in direction of the *i-th* degree of freedom from a unit force applied in the direction of *j-th* degree of freedom. The flexibility matrix has one remarkable property – if you select in it the blocks corresponding to the deformations of individual nodes, form a common (truncated) matrix from these blocks and calculate its inverse matrix, then the matrix thus obtained is nothing more than the stiffness matrix of the entire system reduced to the selected nodes.

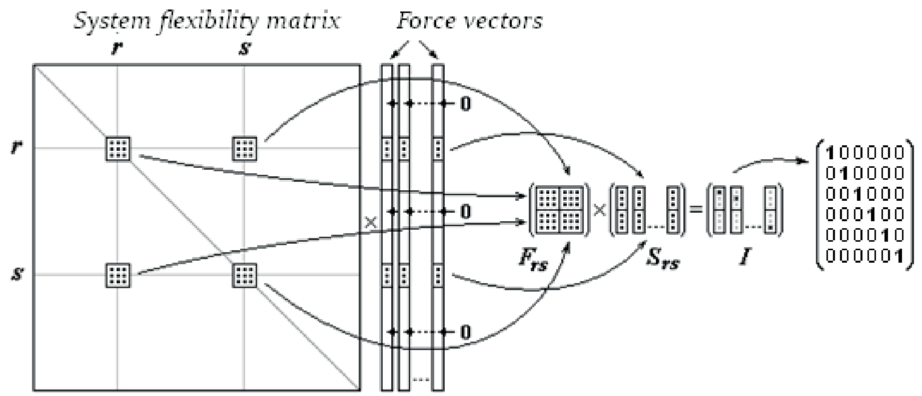

Figure 2. Formation of a truncated flexibility matrix and its transformation into a stiffness matrix of the entire system, reduced to the selected nodes

Indeed, let us introduce a vector of forces whose components for all degrees of freedom, except those corresponding to the selected nodes, are equal to zero (Fig. 2, first column of force vectors). We select the components corresponding to the selected degrees of freedom (in the figure – degrees of freedom related to nodes r and s) in such a way that, when multiplied by the flexibility matrix for the first degree of freedom, related to the selected group of nodes, we obtain a value of generalized displacement equal to 1, and for all others – 0 (Fig. 2, first column of matrix ***I***). Obviously, the force vector selected in this way is nothing other than the reactions in the restrictions imposed on the nodes r and s from a single displacement of the first of these reactions, provided that all other nodes, except for those actually fixed, are free (the force vector is defined in such a way that the reactions in these reactions are zero).

Continuing this procedure for all other restrictions of the truncated system, we arrive at the obvious equality:

$$\boldsymbol{F}_{rs} \cdot \boldsymbol{S}_{rs} = \boldsymbol{I}, \tag{5}$$

where $\boldsymbol{F}_{rs}$ is a truncated flexibility matrix formed from cells of the general flexibility matrix related to selected nodes, $\boldsymbol{S}_{rs}$ – stiffness matrix of the entire system reduced to selected pair of nodes, $\boldsymbol{I}$ unit matrix. Hence, the required stiffness matrix of the system reduced to the selected nodes is determined by the equality

$$\boldsymbol{S}_{rs} = \boldsymbol{F}_{rs}^{-1} \tag{6}$$

Thus, in the presence of flexibility matrix for the entire system, it is possible to select in it the blocks associated with a specific element and form a stiffness matrix reduced to the selected nodes. The next step is to subtract the unit reactions of the tested element $\boldsymbol{S}_M(\alpha_0)$ from the given stiffness matrix $\boldsymbol{S}_{rs}$ and substituting instead the matrix of the same element $\boldsymbol{S}_M(\alpha)$, but depending on the parameter α – eigenvalue of the boundary value problem. But here everything is being hampered to the very matrix of flexibility of the entire system ***S.*** Firstly, the system's flexibility matrix is completely filled, which causes problems with storing the matrix, and secondly, even taking into account the symmetry of the flexibility matrix, the costs of its formation may turn out to be incommensurate with the capabilities of modern household computers. However, the peculiarities of the problem allow us to solve this problem with very little computational effort.

4. FORMATION OF THE FLEXIBILITY MATRIX OF THE ENTIRE STRUCTURE

First, we note that, despite the fact that the flexibility matrix is completely filled, all the matrix blocks of interest to us will be located within the same band structure as the original stiffness matrix. Secondly, since the flexibility matrix (like the stiffness matrix) is symmetrical, it is sufficient to form, for example, only the elements lying below or above the main diagonal. Formally, the procedure for constructing a flexibility matrix is reduced to solving a system of equations

$$\boldsymbol{S} \cdot \boldsymbol{F} = \boldsymbol{I}, \qquad (7)$$

where $\boldsymbol{S}$ and $\boldsymbol{F}$ are the stiffness and flexibility matrices of the entire system, $\boldsymbol{I}$ is the identity matrix. To calculate the flexibility matrix $\boldsymbol{F}$, we will consider equation (7) as a system of n equations with n columns of right-hand sides. The i -*th* column of the right-hand sides contains a unit in the i-*th* position. All other elements of the i-*th* column initially contain zeros. Let us divide the procedure for solving equation (7) into 3 stages: the first is the reduction of the stiffness matrix $\boldsymbol{S}$ to a triangular view using the Gauss method; the second is a forward Gaussian sweep on the matrix $\boldsymbol{I}$; third – Gaussian inverse sweep on matrix $\boldsymbol{I}$. Figure 3 on the left shows the stiffness matrix of the system, reduced to a triangular form by a forward Gaussian sweep. On the right is the process of converting the identity matrix into a flexibility matrix.

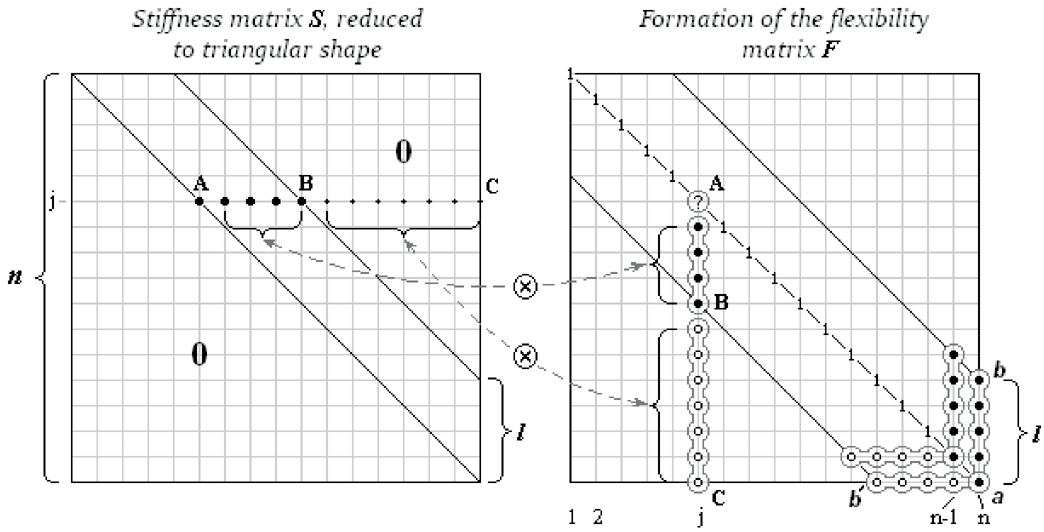

Figure 3. The process of forming the system flexibility matrix

As a result of a direct sweep on the matrix $\boldsymbol{I}$ the entire triangle below the main diagonal will be filled, and the zeros above the main diagonal and the ones on the main diagonal will remain in their original state. Let's consider the reverse sweep procedure using the example of the j-*th* column of the formed flexibility matrix (Fig. 3). Let, as a result of the backward procedure, all elements of the j-*th* column lying below the main diagonal were calculated. To find the diagonal element (indicated by the sign '**?**'), you need to multiply the previously calculated elements of the j-*th* column (**A,C**] of the generated flexibility matrix by the elements of the j-*th* row (**A,C**] of the stiffness matrix. Then subtract the resulting amount from unity (element **A** matrix located on the right) and divide the result by the diagonal element of the stiffness matrix (element **A** matrix located on the left). Obviously, the values of the previously calculated elements of the j-*th* column lying below point **B**, located on boundary of the band structure of the flexibility matrix, will not affect the value of the diagonal element **A** of the flexibility matrices, since all of them will be multiplied by the zeros of the j-*th* row of the stiffness matrix, located above the width of the band. Similarly, when forming all elements of the j-*th* columns located above the main diagonal, elements located below element **B**, will also not affect their values in any way for the same reason. Thus, the procedure for reverse sweep by the j-th column of matrices $\boldsymbol{I}$ can be written as follows:

$$f_{i,j} = \begin{cases} \dfrac{1}{s_{i,i}}\left(1 - \sum\limits_{k=1}^{l_i-1} s_{i,i+k} f_{i+k,i}\right), & i = j, \\ -\dfrac{1}{s_{i,i}} \sum\limits_{k=1}^{l_i-1} s_{i,i+k} f_{i+k,j}, & i = j-1, j-2, \ldots, 1 \end{cases} \tag{8}$$

where l_i – bandwidth of the i-th row. This means that there is no need to either calculate or store flexibility matrix elements located outside the bandwidth. Thus, if there is a way to calculate the elements of the j-th columns of the flexibility matrix located within the width of the band below the diagonal element, without preliminary calculation of the elements located outside the band, then the problem of forming a flexibility matrix will be solved.

The following algorithm solves this problem. We perform the first step of the algorithm – reducing the matrix $\boldsymbol{S}$ to triangular form using the Gaussian procedure**.** The second step is a direct sweep through the matrix $\boldsymbol{I}$**.** But, since the last column of matrix $\boldsymbol{I}$ remains unchanged after the forward sweep (recall that the last n-th column of matrix $\boldsymbol{I}$ contains the only non-zero element – one in cell $[n, n]$), then by omitting the forward sweep, you can immediately perform a reverse sweep along this column. And, since the elements of matrix $\boldsymbol{F}$ outside the width of the band are of no interest to us, the reverse sweep can be made not to the entire height of the matrix, but only within the width of the band – section $[\boldsymbol{ab}]$ (see Fig. 3). Since the flexibility matrix is symmetric, we reflect the elements of the $[\boldsymbol{ab}]$ column onto the $[\boldsymbol{ab}']$ row. Let's move on to the $(n$-1)-th column of matrix $\boldsymbol{F}$**.** Since the subdiagonal element in this column is already known, we immediately perform the reverse sweep, starting with the diagonal element and also within the width of the band. We reflect the calculated elements of the $(n$-1)-th column onto the $(n$-1)-th row. Continuing this procedure, we successively fill in the band structure of the matrix $\boldsymbol{F}$**.** The algorithm can be constructed in such a way that it will not be necessary to fill the subdiagonal part of the matrix $\boldsymbol{F}$**.** Therefore, the required external memory for storing the required data of matrix $\boldsymbol{F}$ no more than what is needed to store the band structure of the system's stiffness matrix.

To avoid reloading the rows of the factorized triangular matrix $\boldsymbol{S}$, the elements of the matrix $\boldsymbol{F}$ should be calculated in the sequence shown in Figure 4.

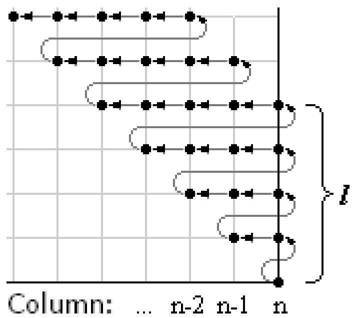

Figure 4. Sequence of formation of elements of the flexibility matrix

The coefficients of the matrix $\boldsymbol{F}$ are found using recurrent formulas.

$$\begin{cases} f_{i,i+m} = -\dfrac{1}{s_{i,i}}\left(\sum\limits_{k=1}^{m} s_{i,i+k} f_{i+k,i+m} + \sum\limits_{k=m+1}^{l_i-1} s_{i,i+k} f_{i+m,i+k}\right), & m > 0, \\ f_{i,i} \quad = \dfrac{1}{s_{i,i}}\left(1 - \sum\limits_{k=1}^{l_i-1} s_{i,i+k} f_{i,i+k}\right), & m = 0. \end{cases} \tag{9}$$

Index i is go over through the sequence $i = n, n-1, \ldots, 1$, and for each value of index i index m runs through values $m = l_i - 1, l_i - 2, \ldots, 0$. Table 1 shows the number of operations required to reduce a matrix of order n and bandwidth l to triangular form, and the number of operations required to form the band structure of the inverse matrix

Table 1. Number of operations

Operation	Direct Gaussian move	Forming the tape inverse matrix
Division	$nl-\frac{l(l-1)}{2}$	$nl-\frac{l(l-1)}{2}$
Multiplication	$\frac{nl(l-1)}{2}-\frac{l(l-1)(2l-1)}{6}$	$nl(l-1)-\frac{l(l-1)(2l-1)}{3}$
Subtraction	$\frac{nl(l-1)}{2}-\frac{l(l-1)(2l-1)}{6}$	$nl(l-1)-\frac{l(l-1)(2l-1)}{3}$

As you can see, the number of operations for forming a band of the inverse matrix is approximately 2 times greater than the number of operations for the Gaussian elimination procedure, which is much more efficient than forming a system stiffness matrix and processing it according to Gaussian for each element separately.

After the "band" of the system's flexibility matrix $\boldsymbol{F}$ is formed follow the procedure described above. For each rod, we "pull out" from the flexibility matrix the cells corresponding to the attachment points of the rod and form a truncated flexibility matrix of the rod attachments $\boldsymbol{F}_{sk}$. This is a relatively small matrix – in the case of a spatial system, it is a 12×12 matrix. We calculate its inverse matrix – the stiffness matrix of the system reduced to the nodes of the tested rod $\boldsymbol{S}_{sk}$. Since the initial matrix of the system is formed taking into account the single reactions of the tested rod, we subtract the rod's own stiffness matrix $\boldsymbol{S}_M(\alpha_0)$ from the given matrix. We add to the given stiffness matrix the rod's own stiffness matrix as a function of the parameter α – $\boldsymbol{S}_M(\alpha)$. We find the eigenvalues of the parameter α. For each value of α we construct a mode of buckling. Based on the mode, we determine relative to which principal axis of the section the bending occurs according to this mode. We continue the search for eigenvalues until we find the smallest eigenvalue for each principal axis of the section. Using the formula $\mu = \pi/\alpha$ we determine the coefficient of the effective length of the rod relative to both principal axes of the section.

5. EXAMPLE

The algorithm described above was successfully implemented on the SELENA computing complex [32]. As an example, consider the flat frame shown in Figure 5. I-section columns HE 320C (**A** = $2.3695\cdot10^{-2}$ m^2, **I** = $4.8711\cdot10^{-4}$ m^4) . Crossbars – box section RVS 300×300×15 (**A** = $1.6521\cdot10^{-2}$ m^2, **I** = $2.1959\cdot10^{-4}$ m^4). The elastic modulus is taken equal to **E** = $2.0594\cdot10^{11}$ N/m^2.

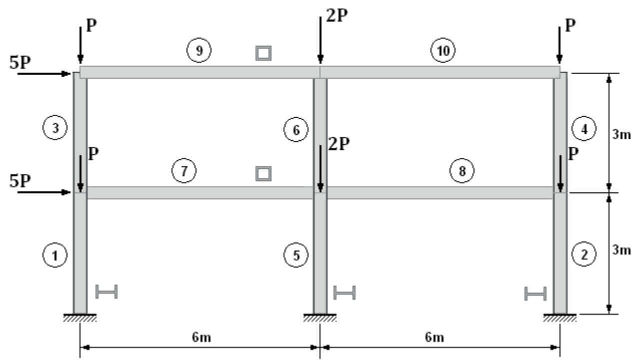

Figure 5. Example of a flat frame

This flat frame structure loses stability at a force value P equal to $1.005\cdot10^{7}$ N. Figure 6 shows the curves of changes in the values of the effective length coefficient μ depending on the load intensity P for six columns calculated in accordance with the described algorithm.

Let's perform "manual" testing of the algorithm using column No.5 as an example (in Figure 7a this element is designated **AB**). For simplicity, we will assume that there is no external load.

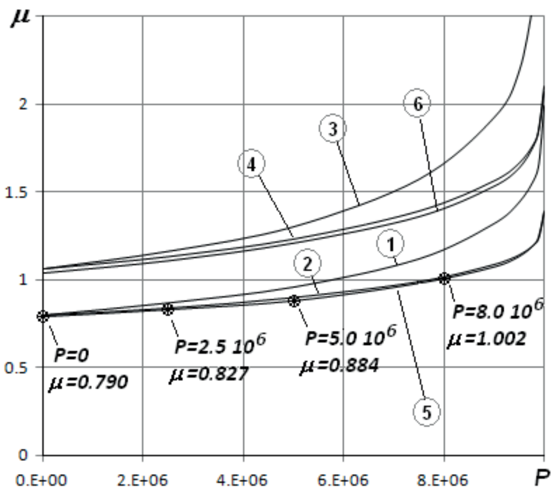

Figure 6. Graphs of the dependence of the coefficient of the effective length μ of columns on the load level P

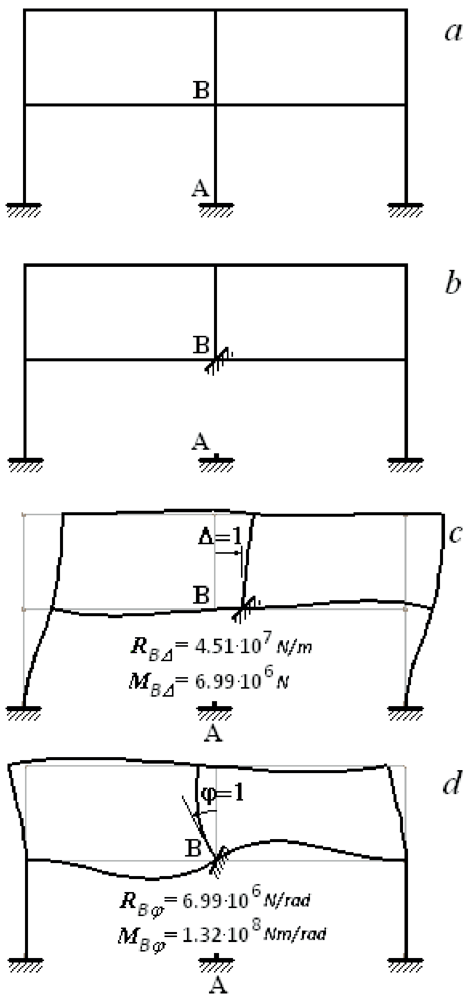

Figure 7. Example of "manual" check for element No.5.

Element No.5 is excluded from the design diagram and constraint is added at node **B** (node **A** is initially fixed, Fig. 7*b*). We give node **B** a unit horizontal displacement $\Delta = 1$ and determine the reactions $R_{B\Delta}$ and $M_{B\Delta}$ (Fig. 7*c*). We rotate node **B** by an angle $\varphi = 1$ and determine the reactions $R_{B\varphi}$ and $M_{B\varphi}$ (Fig. 7*d*□ □□□We add our own unit reactions of rod No.5 as a function of the parameter α (see formulas (3,4)).As a result, the system stiffness matrix reduced to nodes **A** and **B** will have the form

$$\boldsymbol{S} = \begin{pmatrix} R_{B\Delta} + \dfrac{12EI}{l^3}\dfrac{\left(\dfrac{\alpha}{2}\right)^3}{3\left(\mathrm{tg}\dfrac{\alpha}{2} - \dfrac{\alpha}{2}\right)} & R_{B\varphi} + \dfrac{6EI}{l^2}\dfrac{\left(\dfrac{\alpha}{2}\right)^2 \mathrm{tg}\dfrac{\alpha}{2}}{3\left(\mathrm{tg}\dfrac{\alpha}{2} - \dfrac{\alpha}{2}\right)} \\ M_{B\Delta} + \dfrac{6EI}{l^2}\dfrac{\left(\dfrac{\alpha}{2}\right)^2 \mathrm{tg}\dfrac{\alpha}{2}}{3\left(\mathrm{tg}\dfrac{\alpha}{2} - \dfrac{\alpha}{2}\right)} & M_{B\varphi} + \dfrac{4EI}{l}\dfrac{\dfrac{\alpha}{2}\left(1 - \dfrac{\alpha}{\mathrm{tg}\alpha}\right)}{4\left(\mathrm{tg}\dfrac{\alpha}{2} - \dfrac{\alpha}{2}\right)} \end{pmatrix} . \qquad (10)$$

For element No.5 we have: $\frac{12EI}{l^3} = \frac{12 \cdot 2.0594\ 10^{11} \cdot 4.8711\ 10^{-4}}{3^3} = 0.4458\ 10^8 N/m.$

$\frac{6EI}{l^2} = \frac{12 \cdot 2.0594\ 10^{11} \cdot 4.8711\ 10^{-4}}{3^2} = 0.6688\ 10^8 N.$

$\frac{4EI}{l} = \frac{12 \cdot 2.0594\ 10^{11} \cdot 4.8711\ 10^{-4}}{3} = 1.3375\ 10^8 Nm.$

Now it remains to find a parameter value α that satisfies the requirement $\det(\boldsymbol{S}(\alpha)) = 0$. The zero value of the determinant is reached at $\alpha = 3.977$, which exactly corresponds to the value $\mu = \frac{\pi}{\alpha} = 0.790$ found by the program at zero load (see Fig. 6, point $P = 0$).

Figure 8 shows graphs of changes in the function $\det(\boldsymbol{S}) = f(\alpha)$ at different load levels P. The results are summarized in Table 2. Based on the found values of zero points of function $f(\alpha)$ (Fig. 8) the coefficient μ values corresponding to four load levels were determined and superimposed on the graph $\mu = \mu(P)$ for element No.5 (Fig. 6). As you can see, all the shown marks fall exactly on the curve calculated in accordance with the algorithm.

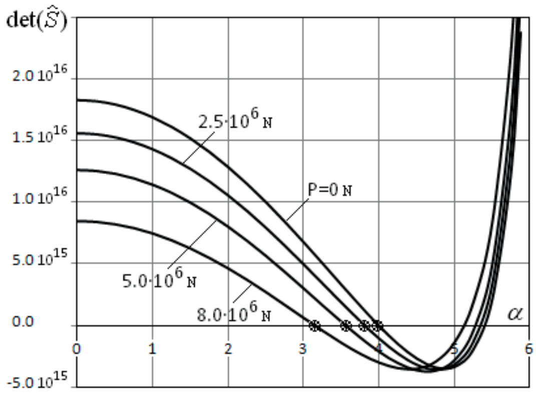

Figure 8. Graph of the function $det(S) = f(\alpha)$ for rod ***AB*** *at various levels of external load*

Table 2. Coefficient α at 4 levels of external load

P (N)	$R_{B\Delta}$ (N/m)	$M_{B\Delta}$ $R_{B\varphi}$ (N)	$M_{B\varphi}$ (Nm/rad)	α	μ
0	$4.507 \cdot 10^7$	$6.993 \cdot 10^6$	$1.316 \cdot 10^8$	3,977	0.790
$2.5 \cdot 10^6$	$3.982 \cdot 10^7$	$7.231 \cdot 10^6$	$1.161 \cdot 10^8$	3.799	0.827
$5.0 \cdot 10^6$	$3.386 \cdot 10^7$	$7.166 \cdot 10^6$	$0.973 \cdot 10^8$	3,554	0.884
$8.0 \cdot 10^6$	$2.473 \cdot 10^7$	$5.278 \cdot 10^6$	$0.636 \cdot 10^8$	3.137	1.002

Fig. 9 shows the design model of a 4-story 4-span frame. All rods are of the same cross-section. The floors and the ceiling are loaded with the same distributed load.

The graphs in Fig. 10 show the curves of the buckling length coefficient μ dependence on the load level for columns (A, B, C), calculated programmatically based on the algorithm described above. The load level in relation to the critical state is plotted on the abscissa scale. The numbers indicate the floor numbers.

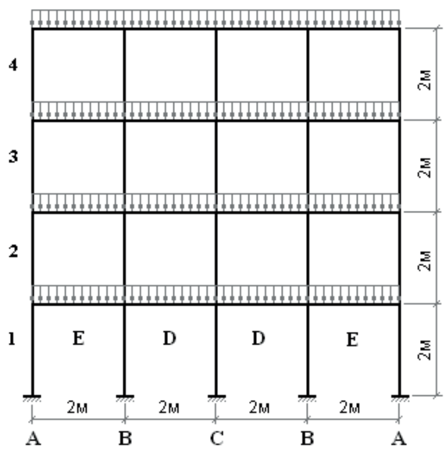

Figure 9. Design model of a 4-storey 4-span frame

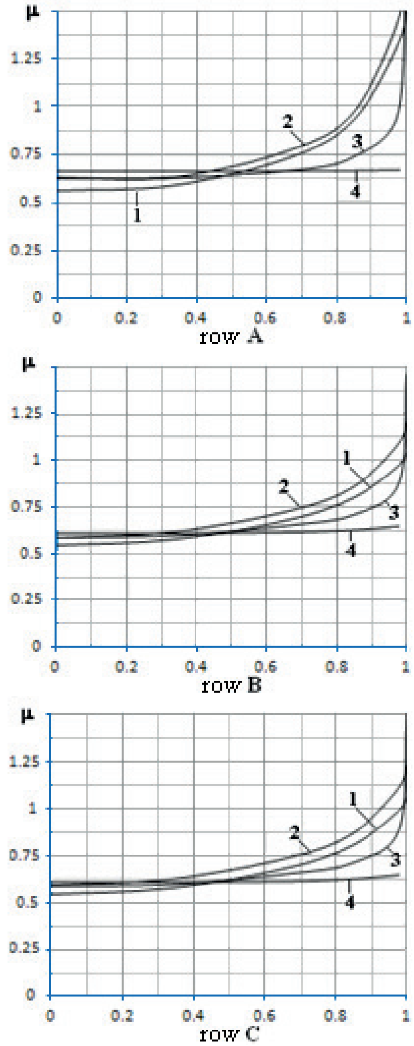

Figure 10. Graphs of the dependence of the coefficient μ on the load level

Table 3 shows the limit values of the coefficient μ corresponding to the critical state of the structure. Note that these limit values correspond to the “standard” calculation scheme of analysis of overall stability, when searching for a critical state, the load increase in all elements of the system proportional to the same parameter.

Table 3. Values of the coefficient μ when the load reaches a critical value

Floor	A	B	C
4	3.141	2.055	2.131
3	2.192	1.464	1.501
2	1.785	1.197	1.225
1	1.545	1.037	1.064

It is somewhat more difficult to check the algorithm works if the acting load does not reach the critical value. To do this, it is necessary to find a way to fix the stress state of the structure at a certain level (below the critical value), and vary the force in one tested rod, bringing it to a loss of stability. Of the programs known to us, only the SCAD software package can provide such an possibility [33]. We are talking about the method of "additional loading of passive elements", described in the work [8]. The program allows, having formed a calculated combination of loadings for performing the overall buckling analysis, to fix individual loadings, and to carry out "additional loading" on the remaining loadings.
In order to test the algorithm operation, the following scheme was implemented - a design combination was created consisting of two loadings. The first loading included a distributed load, as shown in Figure 9. This loading was included in the design combination with a combination factor of 0.9 and was fixed. The second loading consisted of two unit forces applied to the ends of the tested rod and directed towards each other, thus simulating its compression. Variation was allowed only for the second loading. Based on the found critical force, the coefficient of the buckling length of the rod was calculated using the formula $\mu_{test} = \frac{\pi}{l}\sqrt{\frac{EI_{test}}{N_{1,test}+N_{2.test}}}$, where $N_{1,test} -$ is the force in the tested rod from the first loading, and $N_{2,test} -$ is the critical force in the same rod from the second loading. All column elements in rows A, B and C were tested. Table 4 shows the results of the experiment.

Table 4. Values of the coefficient μ when the load reaches 0.9 of the critical value

Floor	A	B	C
4	$\frac{0.668}{0.668}$	$\frac{0.636}{0.635}$	$\frac{0.632}{0.631}$
3	$\frac{0.784}{0.784}$	$\frac{0.748}{0.747}$	$\frac{0.749}{0.747}$
2	$\frac{1.082}{1.083}$	$\frac{0.930}{0.929}$	$\frac{0.935}{0.934}$
1	$\frac{1.039}{1.039}$	$\frac{0.859}{0.857}$	$\frac{0.867}{0.865}$

The numerator of each fraction contains the value of the coefficient μ obtained by the SCAD program, and the denominator contains the value obtained by the described algorithm implemented in the SELENA program. As can be seen, the results are almost identical.

6. CONCLUSION

The speed of the algorithm described above is ensured by the fact that, as has been shown, the boundary conditions at both ends of a particular element can be obtained by inverting a very small fragment of the overall flexibility matrix of the structure. In turn, it is shown that to solve the problem, not the entire flexibility matrix is needed, but only that part of it that lies within the band structure of the system's stiffness matrix. An effective method for forming the required flexibility matrix band is proposed. Experiments show that the time required to calculate the effective lengths of all structural elements is comparable to the time required for one static calculation per one load. And the fact that only the band structure of the flexibility matrix is used does not impose excessive demands on the use of external memory.
Some notes on using the algorithm. Since the stiffness of a structure depends not only on the geometric characteristics of the elements, but also on their stress state, when preparing the initial stiffness matrix, the unit reactions of the elements must be included taking into account the longitudinal forces acting in them. To ensure a certain safety factor of structural strength, it can be recommended to increase the values of forces from the design combination of loads by 15-20%. A more strong option – perform a preliminary search for the most unfavorable combinations of loads and for each element set the largest value of longitudinal force over the entire sample of possible load combinations. This seems all the more appropriate since calculators, as a rule, do not recalculate the effective lengths of bars for various combinations of load cases. And some overestimation of longitudinal forces in

individual rods only affects the boundary conditions of neighboring rods, which will contribute to a certain robustness of stability. In this case, it is necessary to ensure that the given combination of forces does not take the structure beyond the limits of loss of overall stability. Considering that in reality, in the vast majority of cases, structures have a significantly larger robustness of overall stability, such a cautious approach would provide a guaranteed robustness of strength without excessive waste of material.

The algorithm, in the form in which it is presented in this work, cannot be applied to compound rods and rods with intermediate nodes. With a little modification this shortcoming can be eliminated. The authors hope that they will have the opportunity to continue work in this direction.

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