DOI:10.22337/2587-9618-2025-21-1-39-52

# DEFLECTION AND AXIAL FORCE IN GEOMETRICALLY NON-LINEAR BEAM WITH PINNED SUPPORTS

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**Abstract:** The problem of beam bending for large deflections is described in general. The nonlinear beam theory is considered for a simply-supported beam subjected to uniform load. The governing equations for displacements of this beam are derived. Numerical method for solving the governing equations is proposed. Convergence of the numerical method is studied. Numerical results are shown in the form of figures and formulas. These results suggest that the deflection predicted by the nonlinear theory at a specific point can be expressed solely as a function of the linear deflection at the same point. It is also shown how the axial force in the beam depends on the nonlinear deflection. Analytical expression for the axial force in the beam for small deflections is derived without solving the differential equation. For larger deflections, another representation for the axial force is obtained in terms of auxiliary functions that are defined only in terms of nonlinear and linear deflections. Comparison of the present results with the ABAQUS results is given. It is shown that the present theory can quite accurately predict the deflection and axial force in the beam for large deflections. However, the product of the axial force by the cosine of the slope angle at the support rather than the axial force itself will be a more accurate estimate of the horizontal support reaction.

**Keywords:** large deflection, axial force, nonlinear beam theory, finite difference method, convergence of iterative method, ABAQUS, nonlinear differential equations, principal of virtual work

# ПРОГИБ И ОСЕВАЯ СИЛА В СВОБОДНО-ОПЕРТОЙ ГЕОМЕТРИЧЕСКИ-НЕЛИНЕЙНОЙ БАЛКЕ

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**Аннотация:** В общих чертах описана задача об изгибе балки при больших прогибах. Теория нелинейной балки рассматривается для свободно опертой балки, находящейся под действием равномерной нагрузки. Выведены определяющие уравнения для перемещений этой балки. Предложен численный метод решения определяющих уравнений. Исследована сходимость численного метода. Численные результаты представлены в виде рисунков и формул. Эти результаты позволяют предположить, что прогиб, предсказанный нелинейной теорией в конкретной точке, может быть выражен исключительно как функция линейного прогиба в той же точке. Также показано, как осевая сила в балке зависит от нелинейного прогиба. Аналитическое выражение для осевой силы в балке при малых прогибах получено без решения дифференциального уравнения. Для больших прогибов получено другое представление осевой силы через вспомогательные функции, которые зависят только от нелинейных или линейных прогибов. Приведено сравнение настоящих результатов с результатами по программе ABAQUS. Показано, что настоящая теория может довольно точно предсказать прогиб и осевую силу в балке при больших прогибах. Однако произведение осевой силы на косинус угла наклона на опоре, а не сама осевая сила, будет все же более точной оценкой горизонтальной реакции опоры.

**Ключевые слова:** большие прогибы, осевая сила, нелинейная теория балок, метод конечных разностей, сходимость итерационного метода, программа ABAQUS, нелинейные дифференциальные уравнения, принцип виртуальной работы

## 1. INTRODUCTION

In this paper we analyze a beam of length $L$ subjected to the uniformly distributed load $q$. The beam is simply supported at its ends and also constrained from the axial movement at the supports. When dealing with thin metal sheets, we often have the situation when the thickness of the sheet $t$ is small compared to the distance between the supports. In this case, it becomes important to take into account possible large deflections of the beam, which may be comparable with the thickness $t$ and even larger than $t$. Denote the deflection of the beam by $v$.

The elementary beam theory allows us to account for these large displacements by incorporating axial strain of the beam into the equations. This strain is represented as the sum of the usual term $\frac{du}{dx}$ that is linearly dependent on the axial displacement $u$, and the term $\frac{1}{2}\left(\frac{dv}{dx}\right)^2$ that depends on the vertical displacement $v$ nonlinearly. This approach is used in almost all previous investigations [1-11].

The axial strain naturally leads to the existence of the axial force $N$. For the theory in its simplest form, the axial force $N$ is assumed constant along the length of the beam and it is also equal to the horizontal reaction at the supports. This assumption is very accurate only for sufficiently small displacements.

Timoshenko and Woinowsky-Krieger [1,2] derived exact solutions for beams with various supports. In particular, for a beam with pinned supports they obtained a nonlinear equation that can be solved for the axial force $N$

$$\left(\frac{Et^4}{qL^4}\right)^2 = \frac{135}{16}\frac{\tanh s}{s^9} + \frac{27}{16}\frac{\tanh^2 s}{s^8} - \frac{135}{16s^8} + \frac{9}{8s^6}$$

where

$$s^2 = \frac{3NL^2}{Et^3}.$$

Here $E$ is the Young's modulus.

For faster calculations various approximate expressions were also proposed [1,3]. If we set

$$\alpha = \frac{384}{5}\frac{1}{\pi^2}\frac{N}{qL}\frac{t}{L}\left(\frac{v_{lin}(L/2)}{t}\right),$$

then the axial force $N$ (and subsequently, the horizontal reaction) can be found as a root of the cubic equation

$$\frac{1}{3}\alpha(1+\alpha)^2 = \left(\frac{v_{lin}(L/2)}{t}\right)^2,$$

where $v_{lin}$ is the known displacement predicted by the linear theory, evaluated here at the center of the beam, $x = L/2$.

For larger displacements, more accurate formula for calculation of the horizontal support reaction $R_x$ is desired. It can be obtained by using the fact that the axial force will vary along the length of the beam and the support reaction is equal only to the axial force at the center of the beam.

In this paper, we will obtain certain estimates for the axial force that are very simple and don't require solutions of any nonlinear equations. The exact solution of the differential equation is also not required. In addition, we will obtain a simple upper bound for the axial force.

In our governing equations, we still assume that the axial force $N$ is constant along the length (to simplify the equations), but after finding $N$, the deflection $v$, the beam's slope $v'$ at the support, we will be able to estimate the horizontal support reaction more accurately by evaluating the product of the axial force and the cosine of the slope angle at the support, i.e., $R_x = N\cos v'$. By comparing $R_x$ with ABAQUS software calculations, we have obtained a good match for the horizontal support reaction both for smaller and larger deflections. The value of the axial force $N$ slightly overestimates the actual value of the horizontal support reaction if the displacements become large enough.

## 2. METHODS

### 2.1 Governing equations

Consider a simply-supported beam of length $L$ subjected to the uniformly distributed load $q$. The $x$-axis is directed along the beam and $0 \le x \le L$ (Fig. 1). Let the vertical displacement or deflection of beam's cross-section be denoted by $v = v(x)$. Assume that the height of the cross-section of the beam is equal to $t$, the width is 1. Then the flexural stiffness of the beam $EI$ can be found as

$$EI = E\frac{t^3}{12},$$

where $E$ is the modulus of elasticity, the stiffness in tension or compression is

$$EA = Et.$$

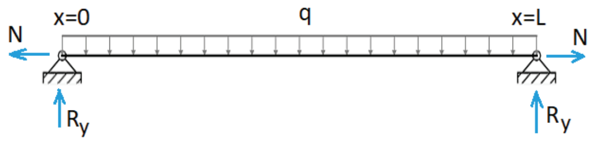


*Figure 1. Beam geometry, applied loads and support reactions*

If $N$ is the internal axial force (assumed constant along the length of the beam), and $M_T$ is the internal bending moment caused by the applied load, the differential equation for the beam's deflection $v$ can be written as

$$EIv'' - Nv = M_T.$$

For uniformly distributed load and for simply-supported beam the bending moment is given by

$$M_T = \frac{q}{2}x(L - x).$$

Due to the supports the boundary conditions for the function $v$ are

$$v(0) = v(L) = 0.$$

Obviously, for the simply supported beam the second derivative of $v$ at the support points is also equal to zero, as the bending moment is equal to zero at the supports

$$v''(0) = v''(L) = 0.$$

Let $u(x)$ be the axial displacement, i.e., the displacement along the $x$-axis of the axis of the beam. It is known from the non-linear strain theory that the longitudinal strain for the points lying on the axis of the beam can be found as

$$\varepsilon_x = u' + \frac{1}{2}(v')^2.$$

Therefore, the axial force is determined from

$$N = EA\left(u' + \frac{1}{2}(v')^2\right),$$

where $EA$ is the stiffness of the beam in tension or compression, $A$ is the cross-sectional area. From the equilibrium in the direction of the $x$-axis, the axial force must be constant, i.e., $\frac{dN}{dx} = 0$, and therefore, the second differential equation for the function $u$ can be written as

$$\frac{d}{dx}\left(EA\left(u' + \frac{1}{2}(v')^2\right)\right) = 0.$$

Since the axial stiffness is constant, this differential equation can be put in another form

$$u'' + v'v'' = 0,$$

where we have used the fact that

$$\frac{d}{dx}\left(\frac{1}{2}(v')^2\right) = v'v''.$$

Boundary conditions for the function $u(x)$ are given by

$$u(0)=u(L)=0.$$

We note that these differential equations can be derived from principal of virtual work. Let $\delta\left(u' + \frac{1}{2}(v')^2\right)$ be the virtual axial strain, and $\delta v''$ be the virtual curvature. Then the internal virtual work can be represented as

$$\int_a^b EIv''\delta v'' dx$$
$$+\int_a^b EA\left(u' + \frac{1}{2}(v')^2\right)\delta\left(u' + \frac{1}{2}(v')^2\right)dx$$

and the external virtual work is

$$\int_a^b q\delta v dx.$$

Equating these works leads to the final form of the equation

$$\int_a^b EIv''\delta v'' dx$$
$$+\int_a^b EA\left(u' + \frac{1}{2}(v')^2\right)\delta\left(u' + \frac{1}{2}(v')^2\right)dx = \int_a^b q\delta v dx.$$

Of course, the virtual displacements must satisfy

$$\delta v(0)=\delta v(L)=0,$$
$$\delta u(0)=\delta u(L)=0.$$

When integrating the expression for the axial force $N$ along the length of the beam, we obtain

$$NL = \frac{EA}{2}\int \left(v'\right)^2 dx$$

since the axial force is constant and

$$\int u' dx = 0.$$

Therefore, the differential equation for the deflection can also be written as

$$EIv''(x) - v\frac{EA}{2L}\int \left(v'\right)^2 dx = \frac{q}{2}x(L-x).$$

**2.2 Numerical solution**

Numerical solutions of two-point boundary value problems by finite difference method have been discussed in [12-15]. Let us describe numerical algorithm for solving the present problem. To solve the given problem numerically, one can proceed with the algorithm that consists of the following steps:

1. Assume first that the axial force $N$ is equal to zero.
2. Solve equation for the deflection $v$. The right-hand side of this equation is the moment caused by transverse loads, namely, $\frac{q}{2}x(L-x)$.
3. Find the first and second derivatives of the deflection $v$ denoted by $v'$ and $v''$.
4. Solve equation for the axial displacement $u$. The right-hand side of the equation becomes equal to $-v'v''$.
5. Find the first derivative of $u$ denoted by $u'$.
6. With the knowledge of the displacements $u$ and $v$, find the new estimate for the axial force as

$$N_{new} = EA\left(u' + \frac{1}{2}(v')^2\right).$$

This force should be independent of $x$ coordinate, and it does not matter at which point this quantity is evaluated.

7. Check for convergence if $N_{new}$ is sufficiently close to $N$. If not, set $N = N_{new}$ (update the axial force) and go to step 2.

Although this algorithm seems to be harmless it actually diverges when the load $q$ gets large enough and deflections grow. To remedy the situation, we propose using in step 7 a different way of updating the axial force, namely,

$$N \leftarrow \frac{N}{2} + \frac{N_{new}}{2}.$$

Thus, we take the average of the previous value of the axial force $N$ and the new value of the force $N_{new}$. With this important correction, this algorithm works well even for large values of the load $q$, and it converges in a smaller number of iterations.

We use standard finite difference scheme to solve differential equations for the displacements $v$ and $u$. We divide the beam length into $n-1$ small intervals, where $n$ is the number of points chosen to be sufficiently large. The length of each interval $h = L/(n-1)$. Then, if we denote $v_i$ and $u_i$ as the displacements evaluated at point $i$, $1 \le i \le n$, the differential equations can be written in finite difference form as

$$EI\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} - Nv_i = M_{Ti}$$
$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = -\frac{v_{i+1} - v_{i-1}}{2h}\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2}.$$

In the first equation the axial force $N$ is assumed constant, chosen as described in the algorithm presented above. To evaluate accurately the second and first derivatives at the boundary points $i = 1$ and $i = n$, we introduce ghost points with the coordinates $x = 0 - h$ and $x = L + h$. We give them numbers $i = 0$ and $i = n+1$, respectively. Then, the system of linear equations for finding $v_i$ or $u_i$ will comprise of $n+2$ equations because there are now $n+2$ points. The first equation in this system of equations will correspond to the finite difference equations evaluated for the left support with $i = 1$, i.e.,

$$EI\frac{v_0 - 2v_1 + v_2}{h^2} - Nv_1 = M_{T1}$$
$$\frac{u_0 - 2u_1 + u_2}{h^2} = -\frac{v_2 - v_0}{2h}\frac{v_0 - 2v_1 + v_2}{h^2}.$$

Similar expressions can be formed for the last equation in the system of linear equations. They are the finite difference equations evaluated at right support point with $i = n$.

### 2.3 Axial force for small displacements

Let us consider the limit of small vertical displacements and obtain estimate for the horizontal support reaction $N$ in this case. In the limit of small deflections, the deflection predicted by the nonlinear theory is approximately equal to the displacement predicted by the linear theory. In the linear theory, the vertical displacement is given by

$$EIv(x) = \frac{q}{2}\left(\frac{x^3 L}{6} - \frac{x^4}{12}\right) - \frac{qL^3}{24}x$$

and the slope can be found as

$$EIv'(x) = \frac{q}{2}\left(\frac{x^2 L}{2} - \frac{x^3}{3}\right) - \frac{qL^3}{24}.$$

The axial force can now be evaluated from the relationship

$$N = \frac{EA}{2L}\int (v')^2 dx.$$

After substitution of the derivative of the linear displacement into this equation and subsequent evaluation of the integral, the axial force can be found as

$$N = \frac{EA}{8}\frac{17}{5040}\left(\frac{qL^3}{EI}\right)^2.$$

Now consider a beam with a cross-section of height $t$, and the unit width. The area of this cross-section $A = t$ and the moment of inertia $I = t^3/12$. In this case, the axial force becomes

$$N = \frac{17}{280}\frac{1}{E}q^2\frac{L^6}{t^5}.$$

and thus the axial force depends on the applied loading quadratically.

It is also known that the deflection $v$ at the center of the beam can be found as

$$\frac{v(L/2)}{t} = \frac{5}{32}\frac{qL^4}{Et^4}.$$

Using this expression, the above formula for the axial force can be transformed to

$$\frac{N}{qL/2} = \frac{136}{175}\frac{v(L/2)}{t}\frac{L}{t} = 0.777\frac{v(L/2)}{t}\frac{L}{t}.$$

This is our estimate for the axial force in the case of small deflections. This formula gives the ratio of the horizontal support reaction force to the vertical support reaction force.

**2.4 Representations using dimensional theory**

Let us obtain representations for the axial force and displacements using dimensional theory. Again consider a beam with a cross-section of thickness $t$ and unit width. Introduce dimensionless (normalized) deflection as

$$w = \frac{v}{t}.$$

Denote the deflections predicted by the linear and nonlinear theories as $w_{lin}$ and $w_{nl} = w$. It is well known that in the linear theory the dimensionless deflection $w$ at the center is given by

$$w_{lin} = \frac{5}{32}\frac{qL^4}{Et^4}.$$

By analyzing numerical results (presented below) we can discover that the nonlinear vertical displacement at a particular point of the beam can be expressed solely in terms of linear vertical displacement evaluated at the same point. Let the function that relates nonlinear and linear displacements be denoted as $g$. Then

$$w_{nl} = g(w_{lin}).$$

This function is, of course, varies from point to point of the beam, but it is remarkable that in this relationship there is no dependence on $L/t$. Let us prove this fact. Recall that our differential equation for the deflection has this form

$$EIv''(x) - v\frac{EA}{2L}\int (v')^2 dx = \frac{q}{2}x(L-x).$$

Let us write this equation in dimensionless form. We introduce dimensionless coordinate $x_1$ as $x_1 = x/L$. Then, again using definitions for the area and the moment of inertia, we can derive

$$\frac{d^2w}{dx_1^2} - 6w\int\left(\frac{dw}{dx_1}\right)^2 dx_1 = 6\frac{qL^4}{Et^4}x_1(1-x_1).$$

Therefore,

$$p = \frac{qL^4}{Et^4} = \frac{32}{5}w_{lin}$$

is a dimensionless parameter that the deflection will depend on. But this parameter can be related to the displacement $w_{lin}$ predicted by the linear theory. Therefore, we proved that upon fixing a point on the beam with the normalized coordinate $x_1$, the nonlinear deflection will depend only on the linear deflection at the same point.

The axial force can also be written in terms of dimensionless deflection $w$ as follows

$$N = \frac{Et}{2}\frac{t^2}{L^2}\int\left(\frac{dw}{dx_1}\right)^2 dx_1.$$

But Young's modulus can be expressed in terms of linear displacement at the center

$$E = \frac{5}{32}\frac{qL^4}{w_{lin}t^4}.$$

Substituting this result into the equation for the axial force we obtain

$$N = \frac{5}{32}\frac{qL}{2}\frac{L}{t}\frac{1}{w_{lin}}\int\left(\frac{dw}{dx_1}\right)^2 dx_1.$$

Thus, the axial force $N$ admits the following representation

$$N = \frac{qL}{2}\frac{L}{t}f(w_{nl}).$$

Here $f$ is some function of nonlinear displacement, the shape of which will be established below, and $qL/2$ is the vertical reaction force $R_y$. This representation for the axial force was obtained by using dimensional analysis on the equation for the deflection $v(x)$. It follows that for a fixed value of nonlinear vertical displacement, the ratio of the axial force to the vertical support reaction force $qL/2$ will depend only on $L/t$, and therefore, will be twice larger for the beam with $L/t = 20$ compared to the beam with $L/t = 10$. Remember, however, that the nonlinear displacement will also depend on $L/t$ in some nonlinear fashion.

Another important observation can be made from the fact that the nonlinear deflection can be expressed solely as a function of the linear deflection. The loading $q$ can be found in terms of the linear deflection at the center as follows

$$q = \frac{32}{5}E\left(\frac{t}{L}\right)^4 w_{lin}.$$

For the same loading $q$ and for the same geometry of the beam, the nonlinear deflection can be found. Using the connection between the linear and nonlinear displacements, this nonlinear deflection is therefore related to the magnitude of the loading by

$$q = \frac{32}{5}E\left(\frac{t}{L}\right)^4 g^{-1}(w_{nl}).$$

Thus, for the nonlinear theory, similar to the linear theory, the loading will also vary as $(t/L)^4$ for a fixed nonlinear deflection. This tells us, for example, that to produce the same nonlinear (normalized) deflection in the beam that is twice longer it is required to apply the loading that is 16 times smaller.

Let us obtain representation for the axial displacement in terms of vertical displacement. Since the first part of the axial force is given by

$$N_1 = EA\frac{du}{dx}$$

we can obtain, using the representation for the axial force, that

$$\frac{du}{dx} = \frac{q}{2E}\left(\frac{L}{t}\right)^2 f(w_{nl}).$$

Thus, using our representation for the loading $q$, we obtain

$$\frac{du}{dx} = \frac{16}{5}\left(\frac{t}{L}\right)^2 f(w_{nl})g^{-1}(w_{nl}).$$

Introduce the dimensionless axial displacement and coordinate $x$ as follows

$$u_1 = \frac{u}{t}, \qquad x_1 = \frac{x}{L}.$$

It is easy to show that

$$\frac{du_1}{dx_1} = \frac{16}{5}\left(\frac{t}{L}\right)f(w_{nl})g^{-1}(w_{nl}).$$

or

$$du_1 = dx_1 \frac{16}{5}\left(\frac{t}{L}\right) f(w_{nl}) g^{-1}(w_{nl}).$$

Therefore, for the same increment in the coordinate $dx_1$ starting from the support (take, for example, quarter of the beam length) and for a fixed value of nonlinear deflection, the increment in the axial displacement will depend only on $t/L$, and therefore, the dimensionless axial displacement will be twice larger for the beam with $L/t=10$ compared to the beam with $L/t=20$. This result will be illustrated below.

## 3. RESULTS AND DISCUSSION

Consider a simply-supported beam of length $L$ subjected to the uniformly distributed load $q$. The height of the cross-section of the beam is denoted by $t$ (thickness), while the width of the cross-section is assumed equal to 1.
Introduce dimensionless (normalized) deflections as

$$w = \frac{v}{t}.$$

Denote the deflections predicted by the linear and nonlinear theories as $w_{lin}$ and $w_{nl}$. In the linear theory the dimensionless deflection $w$ at the center can be found as

$$w_{lin} = \frac{5}{32}\frac{qL^4}{Et^4}.$$

In what follows we investigate beams with two different geometries with $L/t$ equal to 10 and 20. The beams are subjected to the same load $q$. On the following plots the results for the shorter beam with $L/t=10$ are shown with solid lines. For the longer beam with $L/t=20$ the results are shown with dashed lines. The size of the markers correspond (approximately) to the magnitude of the applied load $q$ applied – the marker with the larger size corresponds to the larger magnitude of the load.
Figure 2 shows how the nonlinear displacement (normalized with respect to the thickness of the beam) depends on the linear displacement (also normalized).

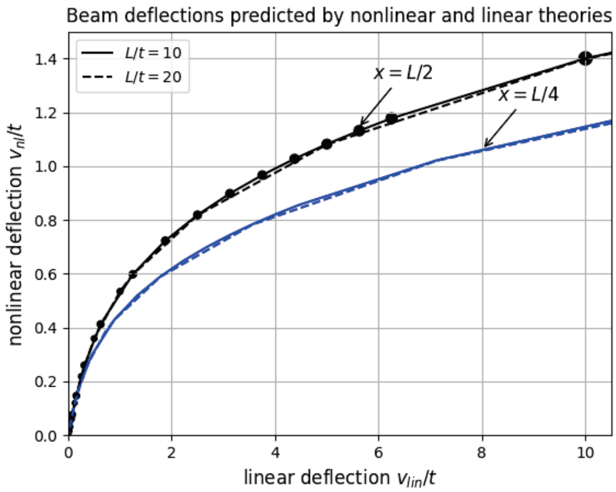


*Figure 2. Dependence of the nonlinear displacement at the center of the beam on the linear displacement*

The displacement is evaluated at the center of the beam, and thus it is maximum, and at a quarter of the total length of the beam. It is clear that the results for the two beam geometries overlap and thus the function that relates nonlinear and linear displacements at a specific point is independent of $L/t$, i.e.,

$$w_{nl} = g(w_{lin}).$$

We see clearly that the nonlinear displacements are significantly lower than the linear displacements. For example, for the linear displacement equal to 1, the nonlinear displacement is about 0.52; for the linear displacement equal to 4, the nonlinear one is only 1.
Figure 3 is central in our presentation. This figure shows by how much the horizontal support reaction $N$ is smaller or larger than the vertical support reaction $qL/2$. Let us call the ratio of

the horizontal support reaction to the vertical reaction as the support reactions ratio. The figure shows how this ratio depends on the value of the normalized nonlinear displacement (at the center of the beam) for two geometries of the beam with $L/t = 10$ and $L/t = 20$. Size of the circles in the figure corresponds approximately to the magnitude of the applied load $q$.

Initially, of course, when the displacements are very small and the load is small, the horizontal support reaction is smaller than the vertical support reaction. It was shown in the previous section that for small loads the support reactions ratio can be well fitted with the function $0.777L/t(v/t)$ where $v/t$ is normalized deflection at the center (linear or nonlinear deflections are about the same). It is seen from the graph that this approximation can be used with good accuracy when the deflection $v/t$ is smaller than 0.2.

The point where the support reactions ratio becomes exactly equal to 1 depends on the ratio $L/t$: for more flexible beams with larger $L/t$ it happens earlier, at smaller loads and at smaller displacements. After this point the horizontal support reaction quickly becomes larger than the vertical support reaction. It is interesting to observe that the point where the support reactions ratio reaches maximum does not depend on $L/t$: it happens when the nonlinear deflection at the center is about 0.59. For this value of nonlinear displacement the support reactions ratio can be evaluated approximately as $0.22L/t$. Therefore, the maximum of the support reactions ratio for $L/t = 20$ is twice larger than for $L/t = 10$.

In addition, the figure shows the ratio of $R_x$, defined as $N\cos(v'(0))$, to the vertical support reaction. It is seen that $R_x$ is about the same as $N$ except for very large displacements when it becomes somewhat smaller than $N$. This implies that the angle of slope of the deflected shape of the beam $v'(0)$ remains small even for large displacements.

We note that both the horizontal support reaction and vertical support reaction always grow with the increase of the load and the nonlinear displacement, but this figure shows that the rate of growth of the horizontal support reaction always changes during the loading. Indeed, for very large displacements the support reactions ratio decreases and again reaches the value of 1 and lower. Let us predict the support reactions ratio for large deflections from the elementary theory of flexible cables. Neglecting bending moment at the center we can obtain from the equilibrium of the half of the beam that this ratio is equal to $L/(4t)1/(v_{nl}/t)$, where $v_{nl}/t$ is normalized nonlinear displacement at the center. Thus, this formula shows that the support reactions ratio for cables will eventually approach zero with the growth of nonlinear displacement, and thus also with the growth of vertical support reaction $qL/2$. This approximation for the support reactions ratio can be used with the good accuracy when $v_{nl}/t$ is larger than 2.

But to use this graph for prediction of the axial force $N$ or support reactions ratio, we must also remember that for different $L/t$ ratios, the nonlinear displacement $v_{nl}/t$ will also be different.

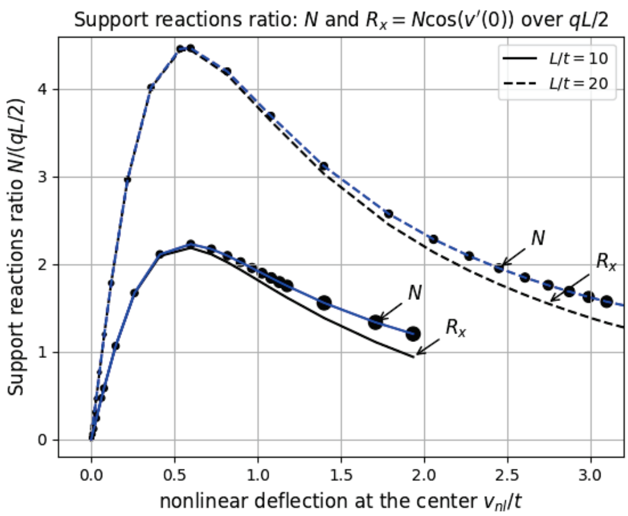


*Figure 3. Ratio of the horizontal reaction force to the vertical reaction force as a function of the nonlinear displacement for two geometries with $L/t$ equal to 10 and 20.*

Figure 4 shows how $f$ depends on the nonlinear displacement evaluated at the center of the beam. The function $f$ is obtained by dividing the axial force $N$ by $qL/2$ and $L/t$, i.e.,

$$f(w_{nl})=\frac{N}{\frac{qL}{2}\frac{L}{t}}.$$

The results shown in Figure 4 do not depend on $L/t$, as expected.

We see that the maximum value of the function $f$ is about 0.22. This maximum is achieved when the nonlinear displacement is 0.59.

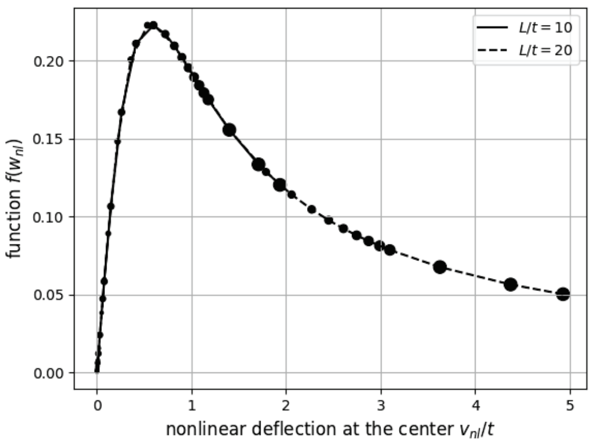


*Figure 4. Dependence of the function $f$ on the nonlinear displacement at the center of the beam*

Using this maximum value, we can always obtain an upper bound for the axial force $N$,

$$N_u = 0.22\frac{qL}{2}\frac{L}{t},$$

although it will not be accurate if the nonlinear deflection is very different from 0.59.

The function $f$ can also be expressed in terms of the linear displacement since we know from Figure 2 how the nonlinear deflection depends on the linear one. This dependence $f(w_{lin})$ is shown in Figure 5. To avoid skewed graph, we plot logarithm of the linear displacement on the horizontal axis of the graph instead of the linear displacement. We clearly see that the maximum of the function $f$ is equal to about 0.22 and it is achieved when the normalized linear displacement is equal to about 1 ($\log_{10} 1 = 0$). Instead of using Figure 5 to find the value of the function $f$ for a given linear deflection, one could use another option for determining the value $f$. Namely, find first the nonlinear displacement from Figure 2, and then determine the value of the function $f$ from Figure 4.

Figure 6 shows maximum horizontal displacement (normalized with respect to thickness) versus vertical displacement for two beam geometries with $L/t$ equal to 10 and 20.

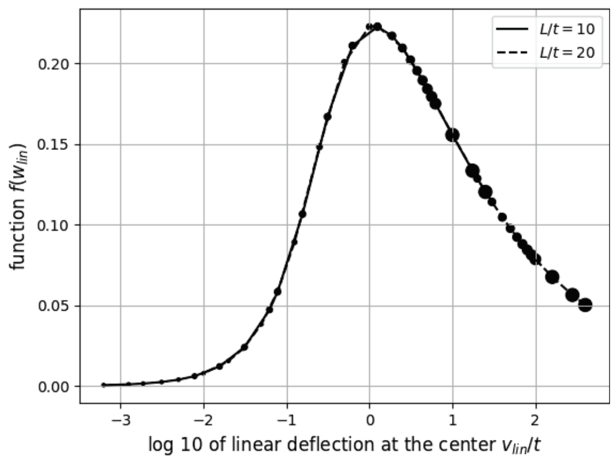


*Figure 5. Dependence of the function $f$ on the logarithm of the linear displacement at the center of the beam*

The vertical displacement is evaluated at the center of the beam, but the maximum of the horizontal displacement occurs at $x = 3/4L$ and $x = 1/4L$. It is seen that the horizontal displacement is much smaller than the vertical displacement. Also, for the same value of the vertical displacement, the horizontal displacement for the beam with $L/t = 10$ is twice larger than that for the beam with $L/t = 20$. Since the horizontal displacement is maximum at $x = 3/4L$ and $x = 1/4L$. it follows that the horizontal displacement has extremums exactly at these points. This fact, however, requires a strict proof.

Comparison of the numerical results obtained using the present technique with the ABAQUS software results is presented in Figure 7.

On the top figure we plot dependence of $R_x$, computed as product $N\cos v'(L)$, on the vertical reaction force $R_y = qL/2$. On the bottom figure we plot the axial force $N$, computed using the present method, versus the vertical reaction force. On both figures we compare these quantities with the horizontal reaction force computed in ABAQUS program $R_x(aba)$.

For convenience, we show absolute (not normalized) quantities for some specific choice of geometrical and physical parameters. But as before, solid line corresponds to $L/t = 10$, and dashed line corresponds to $L/t = 20$.

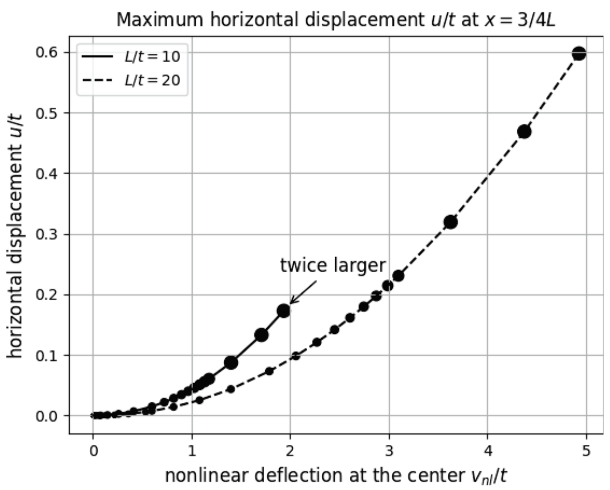


*Figure 6. Maximum horizontal displacement versus maximum vertical displacement*

The horizontal reaction force computed via ABAQUS $R_x(aba)$ is shown with circles for $L/t = 10$ and diamonds for $L/t = 20$. A good match is observed between the results on the top figure, but on the bottom figure we see some discrepancy between the results for larger values of the load $q$. Thus, the axial force multiplied by the cosine of the slope angle at the support $N\cos v'(L)$ is a better estimate for the horizontal reaction force than simply $N$.

Note that this graph shows that the horizontal support reaction always increases with increasing load $q$ although the rate at which the horizontal support reaction grows diminishes.

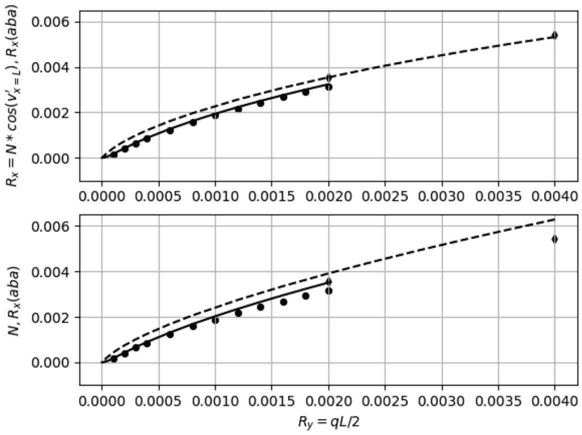


*Figure 7. Comparison of $R_x$ and the axial force $N$, computed using the present method (lines), with the horizontal reaction obtained in ABAQUS (dots). ABAQUS results are shown with markers – circles for $L/t = 10$ and diamonds for $L/t = 20$.*

## 4. CONCLUSIONS

In this paper we have considered a well-known problem of determining deflection and horizontal support reaction for the simply-supported beam constrained from axial movement at the supports and subjected to the uniformly distributed load along the length. The axial strain of the beam included not only the usual term $\frac{du}{dx}$ linearly dependent on the axial displacement $u$, but also the term $\frac{1}{2}\left(\frac{dv}{dx}\right)^2$ depending on the vertical displacement $v$ nonlinearly.

We have presented numerical procedure for solving the governing system of equations for the displacements $u$ and $v$. This procedure is iterative in which on each step the axial force $N$ is updated. In order to ensure convergence of this scheme, we have come up with the rule for updating the axial force $N$, namely, the value of the axial force for the next iteration is taken as the average of the axial force used at the current iteration and the predicted value of the axial force also evaluated at the current iteration. With this update, the iterative scheme was convergent even for large loads and displacements. Convergence was achieved within a small number of iterations.

We have shown on the plots that the nonlinear deflection can be represented as a function of the linear deflection only. Also we have presented the graph for the function $f$ that enables us to evaluate the axial force according to

$$N = \frac{qL}{2}\frac{L}{t}f.$$

The function $f$ can be represented as the function of the nonlinear displacement or as the function of the linear displacement. The maximum of the function $f$ is equal to 0.22, and this allows us to obtain an upper bound on the axial force $N$ for any magnitude of the load $q$:

$$N \le 0.22\frac{qL}{2}\frac{L}{t}.$$

The maximum of the function $f$ is achieved when the normalized nonlinear displacement is equal to about 0.59, and the normalized linear displacement is equal to about 1.

We have also obtained an expression for the axial force in the limit of small displacements and it was shown that for small displacements the axial force depends on the loading quadratically. The support reactions ratio was shown to be a linear function of the vertical displacement. These approximations work well when the normalized deflection at the center does not exceed 0.2. We have also established that when the normalized deflection exceeds 2, the bending moment can be neglected in the beam and the cable approximation can be used to predict the axial force.

By comparing our results with more accurate ABAQUS finite element results, we have observed a very good match for the axial force $N$. However, the horizontal support reaction can be more accurately predicted by using the formula

$$R_x = N\cos v'(0) = N\cos v'(L).$$

This formula becomes more accurate for larger displacements and our calculations allow us to obtain $R_x$ by using this formula since the beam's slope $v'$ is available at each point.

## REFERENCES


1. **Timoshenko S., Woinowsky-Krieger S.** Theory of plates and shells. Mcgraw-Hill Book Company, 1959, 594 p.
2. **Woinowsky-Krieger, S.** The effect of an axial force on the vibration of hinged bars, J. Appl. Mech., 1950, 17, pp. 35–36.
3. **Gibshman E.E.** Proektirovanie metallicheskikh mostov [Steel bridge design]. Moscow, Transport, 1969, 416 p. (in Russian)
4. **Belenya E.I., Baldin V.A., Vedenikov G.S.** Metallicheskie kontstruktsii [Steel structures]. Moscow, Stroyizdat, 1986, 560 p. (in Russian)
5. **Loc V. Tran, Jarkko Niiranen**. A geometrically nonlinear Euler-Bernoulli beam model within strain gradient elasticity with isogeometric analysis and lattice structure applications, Mathematics and Mechanics of Complex Systems, 2020, vol. 8, no. 4.
6. **Paavani D., Aswathy M., Arun C.O., Praveen Krishna I.R.** Analysis of geometrically nonlinear Euler-Bernoulli beam using EFGM, IOP Conf. Series:

Materials Science and Engineering, 2020, 936, 012050.
7. **Worden K., Tomlinson G.R.** Nonlinearity in Structural Dynamics (Detection, Identification and Modelling), Institute of Physics Publishing, Bristol and Philadelphia, 2001.
8. **Lukash P.A**. Osnovy nelineinoy stroitel’noi mekhaniki [Foundations of Nonlinear Structural Mechanics]. Moscow, Stroyizdat, 1978, 204 p. (in Russian)
9. **Reddy J.N.** An Introduction to Nonlinear Finite Element Analysis. Oxford Univeristy Press, 2010.
10. **Jain S., Tiso P., Haller G**. Exact nonlinear model reduction for a von Karman beam: Slow-fast decomposition and spectral submanifolds, Journal of Sound and Vibration, 2018, vol. 423, pp. 195-211.
11. **Mehrpouya M.A., Salehi R., Wong P.J.Y.** A Fast and Accurate Numerical Method for Solving Nonlinear Fourth-Order Boundary Value Problems in the Beam Theory. Axioms, 2024, 13, 757.
12. **Kiusalaas J.** Numerical Methods in Engineering with Python 3. Cambridge University Press, 2013.
13. **Hoffman J.D.** Numerical Methods for Engineers and Scientists. McGraw-Hill, Inc., New York, 1992
14. **LeVeque R.J.** Finite Difference Methods for Ordinary and Partial Differential Equations. Society for Industrial and Applied Mathematics, Philadelphia, 2007.
15. **Butler J.S.** Numerical Methods for Differential Equations with Python. https://johnsbutler.netlify.app.

## СПИСОК ЛИТЕРАТУРЫ

1. **Timoshenko S., Woinowsky-Krieger S.** Theory of plates and shells. Mcgraw-Hill Book Company, 1959, 594 p.
2. **Woinowsky-Krieger, S.** The effect of an axial force on the vibration of hinged bars, J. Appl. Mech., 1950, 17, pp. 35–36.
3. **Гибшман Е.Е.** Проектирование металлических мостов. – М.: Транспорт, 1969. – 416 с.
4. **Беленя Е.И., Балдин В.А., Веденников Г.С.** Металлические конструкции. – М.: Стройиздат, 1986. – 560 с.
5. **Loc V. Tran, Jarkko Niiranen.** A geometrically nonlinear Euler-Bernoulli beam model within strain gradient elasticity with isogeometric analysis and lattice structure applications, Mathematics and Mechanics of Complex Systems, 2020, vol. 8, no. 4.
6. **Paavani D., Aswathy M., Arun C.O., Praveen Krishna I.R.** Analysis of geometrically nonlinear Euler-Bernoulli beam using EFGM, IOP Conf. Series: Materials Science and Engineering, 2020, 936, 012050.
7. **Worden K., Tomlinson G.R.** Nonlinearity in Structural Dynamics (Detection, Identification and Modelling), Institute of Physics Publishing, Bristol and Philadelphia, 2001.
8. **Лукаш П.А.** Основы нелинейной строительной механики. – М.: Стройиздат, 1978. – 204 с.
9. **Reddy J.N.** An Introduction to Nonlinear Finite Element Analysis. Oxford Univeristy Press, 2010.
10. **Jain S., Tiso P., Haller G.** Exact nonlinear model reduction for a von Karman beam: Slow-fast decomposition and spectral submanifolds, Journal of Sound and Vibration, 2018, vol. 423, pp. 195-211.
11. **Mehrpouya M.A., Salehi R., Wong P.J.Y.** A Fast and Accurate Numerical Method for Solving Nonlinear Fourth-Order Boundary Value Problems in the Beam Theory. Axioms, 2024, 13, 757.
12. **Kiusalaas J.** Numerical Methods in Engineering with Python 3. Cambridge University Press, 2013.
13. **Hoffman J.D.** Numerical Methods for Engineers and Scientists. McGraw-Hill, Inc., New York, 1992

14. **LeVeque R.J.** Finite Difference Methods for Ordinary and Partial Differential Equations. Society for Industrial and Applied Mathematics, Philadelphia, 2007.

15. **Butler J.S.** Numerical Methods for Differential Equations with Python. https://johnsbutler.netlify.app.

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