

A PROBABLISTIC APPROACH TO EVALUATION OF THE ULTIMATE LOAD ON FLEXURAL RC ELEMENT ON CRACK LENGTH

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Abstract: The fracture mechanics of concrete and reinforced concrete is a promising direction in the development of methods for reinforced concrete structural elements design and inspection. At the same time, probabilistic methods of design and behavior analysis of structural elements are of particular interest. The article describes a probabilistic approach to load-bearing capacity and reliability analysis of flexural reinforced concrete elements based on the crack length criterion. The functional relationship between the critical stress intensity coefficient of concrete and the design compressive strength of concrete is given. The article presents a method for the reliability analysis of flexural reinforced concrete elements at the operational stage with limited statistical data about the critical stress intensity coefficient of concrete. The ultimate value of the failure probability (or reliability index) should be set for each object individually based on the value of the acceptable risk.

Key words: reliability theory, fracture mechanics, crack length, reinforced concrete beam, reinforced concrete slab, safety

ВЕРОЯТНОСТНЫЙ ПОДХОД К ОПРЕДЕЛЕНИЮ ДОПУСТИМОЙ НАГРУЗКИ НА ИЗГИБАЕМЫЙ ЖЕЛЕЗОБЕТОННЫЙ ЭЛЕМЕНТ ПО КРИТЕРИЮ ДЛИНЫ ТРЕЩИНЫ

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Аннотация: Механика разрушения бетона и железобетона является перспективным направлением в развитии методов расчета железобетонных элементов конструкций. В то же время, особый интерес представляют вероятностные методы расчета и анализа работы несущих элементов строительных конструкций. В работе рассмотрен вероятностный подход к расчету несущей способности и надежности изгибаемых железобетонных элементов по критерию длины трещины. Приведена функциональная зависимость между критическим коэффициентом интенсивности напряжений бетона и расчетным сопротивлением бетона. В статье представлен метод расчета надежности изгибаемого железобетонного элемента на стадии эксплуатации при ограниченной статистической информации о критическом коэффициенте интенсивности напряжений бетона. Предельное значение вероятности безотказной работы (или индекса надежности) следует устанавливать для каждого объекта индивидуально, исходя из значения допустимого риска.

Ключевые слова: теория надежности, механика разрушения, длина трещины, железобетонная балка, железобетонная плита, безопасность

1. INTRODUCTION

Reinforced concrete flexural elements (beams and slabs) are common parts of different structures. The safety of a whole structure

depends from the reinforced concrete beams/slabs safety and reliability. By Eurocode 0 “Basis of structural design”, the reliability - the ability of a structure or a structural member to fulfill the specified requirements, including

the design working life, for which it has been designed. Reliability covers safety, serviceability and durability of a structure and is usually expressed in probabilistic terms. The measure of reliability is the failure probability or safety probability.

As noted in [1], for an adequate description of structural behavior, probabilistic methods must be resorted to. Properly speaking, an element of probability is embodied even in the deterministic approach, which claims to "simplify" the structure by eliminating all aspects of uncertainty. In practice structural reliability (or structural probabilistic design) increasingly is being applied, particularly for situations where quantitative, data-based risk assessment of non-elementary structural or other systems required [2].

Reinforced concrete (RC) flexural elements - beams and slabs are common structural elements in many buildings. Safety assessment of these elements is an important task. K.A.Piradov and N.V. Savickij [3] note that there is no theoretically justified approach for the design of reinforced concrete structural elements with cracks at the moment (reinforced concrete elements, especially without reinforcement prestressing, usually contain cracks at design loads), and current design method (safety factors method or limit state method from 1955) is based on a number of theoretically unsubstantiated empirical coefficients. Fracture mechanics [4, 5, 6] can be successfully applied for design of reinforced concrete elements with cracks. The relationship between fracture mechanics and reliability theory can be a powerful tool for evaluating the structural safety of reinforced concrete elements.

2. METHODS

There are different approaches to limiting the normal crack length in reinforced concrete beams. Some approaches limit the crack length to a percentage of the element's cross-section

height. The research [7] notes that the crack length must not exceed $0.3h_0$, where h_0 – distance from extreme compression fiber to centroid of longitudinal tension reinforcement. Gvozdev A.A. [8] proposed to limit crack length by value $0.5h$, where h – beam cross-section height. The research [9] offers next critical values: $0,7h$, if there is a crack in the middle of the beam span; $0,65h$, if in a third of the span and $0,3h$ if at the support points (shear area).

More objective provisions for limiting the crack length can be obtained from the fracture mechanics equations for concrete and reinforced concrete. Thus, the following dependence is proposed in [10]:

$$l_{cr} = \frac{M^2 \cdot Y_1^2(a_1) \cdot ([2\pi - 1]/2\pi)}{(K_{IC}^b + K_{IC}^s)^2 b^2 h}, \quad (1)$$

where M – bending moment in the beam cross-section; K_{IC}^b – critical stress intensity coefficient of concrete; K_{IC}^s – critical stress intensity coefficient which characterizes the restraining effect of reinforcement on crack growth; b – width of the beam cross-section; h – height of the beam cross-section.

Function $Y_1(a_1)$ have the form

$$Y_1(a_1) = \sqrt{\pi} - a_1 - 1,$$

where $a_1 = a/h$; a – distance from extreme tensile fiber to centroid of longitudinal tension reinforcement.

The parameter K_{IC}^s have the form (for normal crack):

$$K_{IC}^s = \frac{60A_s}{b\sqrt{\pi} \cdot a} \left[\frac{0,93}{\sqrt{1-a_1}} + \frac{1}{\sqrt{1-a_1^2}} - 0,93 \right]. \quad (2)$$

If we limit the crack length to a critical value $l_{cr,ult}$, then the following equation can be derived from equation (1) to evaluate the

bearing capacity (ultimate bending moment M_{ult}) of beam:

$$M_{ult} = \sqrt{\frac{l_{crc,ult} \cdot (K_{IC}^b + K_{IC}^s)^2 b^2 h}{\left(\sqrt{\pi} - \frac{a}{h} - 1\right) \cdot \left(\frac{2\pi - 1}{2\pi}\right)}}. \quad (3)$$

In accordance with the recommendations [10] and SP 63.13330.2018 "Concrete and reinforced concrete structures", the relationship between the critical stress intensity coefficient in concrete and the design compressive strength of concrete can be represented graphically (see Figure 1).

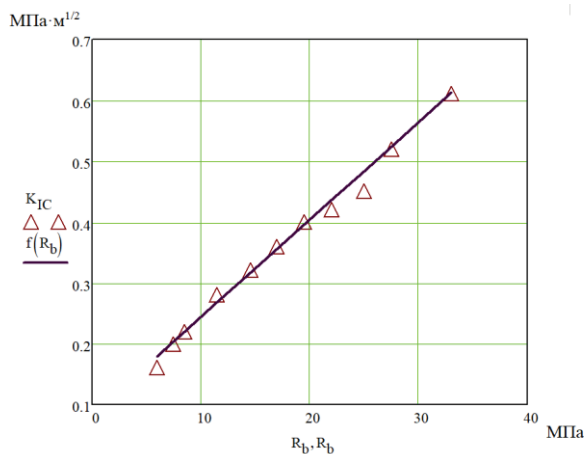


Figure 1. Functional relationship between the critical stress intensity coefficient in concrete and the design compressive strength of concrete.

This dependence also can be approximated as a linear function (in MPa):

$$K_{IC}^b(R_b) = 0.084 + 0.016 \cdot R_b.$$

Thus, having the functional relationship between the critical stress intensity coefficient in concrete and the design compressive strength of concrete it is possible to calculate critical stress intensity coefficient for the existing concrete by determining the compressive

strength (for example, by non-destructive testing).

The problem of assigning the ultimate crack length for a reinforced concrete beam also can be solved. If we take

$$M_{ult} = R_b b x (h_0 - 0.5x),$$

then the ultimate crack length can be expressed as:

$$l_{crc,ult} = \frac{[R_b b x (h_0 - 0.5x)]^2 \cdot [\sqrt{\pi} - a_1 - 1]^2}{(K_{IC}^b + K_{IC}^s)^2 b^2 h} \times \left(\frac{2\pi - 1}{2\pi}\right) \quad (4)$$

where $x = \frac{R_s A_s}{R_b b}$,

R_b and R_s – compressive strength of concrete and tensile strength of reinforcement; A_s – area of nonprestressed longitudinal tension reinforcement.

3. RESULTS AND DISCUSSION

Example 1. The reinforced concrete beam (without prestressed reinforcement) with cross-section dimensions $h=500$ mm and $b=250$ mm. Reinforcement: 5 bars with $\phi 12$ mm ($A_s = 1.231 \cdot 10^{-4}$ m²), with distance from extreme tensile fiber to centroid of longitudinal tension reinforcement $a=40$ mm. Then by eq. (2): $K_{IC}^s = 0,543$ MPa·m². Beam span $l=6$ m. If beam simply supported, then:

$$M = \frac{ql^2}{8}.$$

Figure 2 shows graph of the dependence of the ultimate load q on the required height h of cross section (with $K_{IC}^b = 0,32$ MPa·m²) for different approaches to limiting the crack length in

concrete – $0.3h$, $0.5h$ and $0.7h$ according to example 1 data.

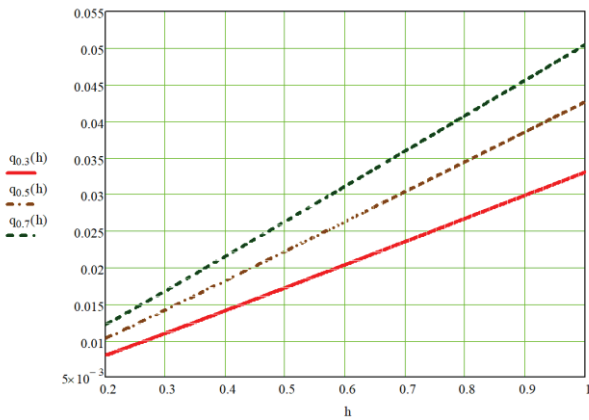


Figure 2. Graphs of dependence of the required height of beam cross-section h and the ultimate load q .

Figure 3 presents the dependence of the ultimate crack length in concrete l_{cr} and design compressive strength of concrete R_b .

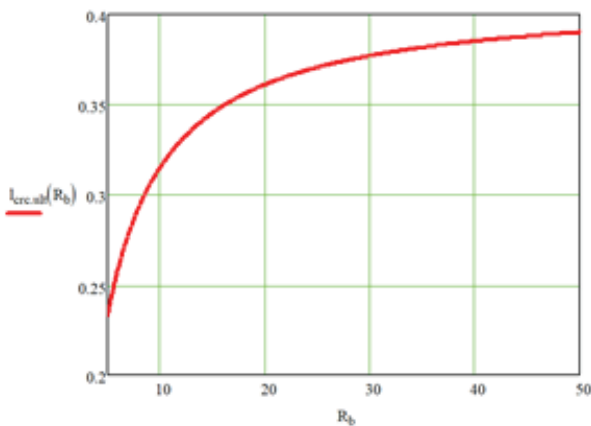


Figure 3. “Ultimate crack length – concrete resistance” diagram.

Figure 3 shows that with a small compressive strength of concrete (10 MPa), the ultimate crack length is 320 mm or $\sim 0.64h$ with an increasing of compressive strength, the ultimate crack length increases to the value $\sim 0.74h$ at 20 MPa and $\sim 0.76h$ at 30 MPa. The ultimate crack length is stabilized to a value of $\sim 0.78h$ next. Critical stress intensity coefficient determined by experimental methods for existing reinforced concrete structural elements. These methods are

often based on the correlation between critical stress intensity coefficient in concrete and the design compressive strength. Let’s consider the problem of estimation the ultimate load q with a given confidence level if critical stress intensity coefficient is random variable with normal distribution. The problem is the simplest problem of the reliability theory [2] with a single random variable, so we present it without additional layouts.

Figure 4 shows the dependence of ultimate load q and K_{IC} variation coefficient C_x at the different significance levels. Data is taken from example 1.

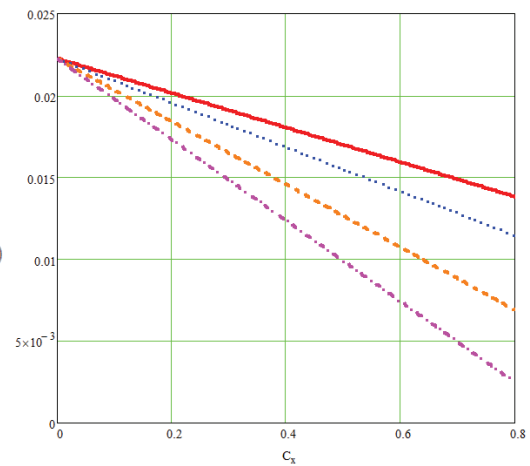


Figure 4. Dependence of ultimate load q and K_{IC} variation coefficient at the different significance levels.

Figure 4 shows that with the increase in the variability of the critical stress intensity coefficient, it is necessary to limit the ultimate load on the RC element to a greater extent in order to ensure the given level of reliability.

The inverse problem can also be solved –the reliability analysis of a reinforced concrete beam by the crack length criterion at the operation stage. A limit state mathematical model for reinforced concrete beam can be presented as:

$$\tilde{q} \leq \frac{l_{cr,ult} \cdot (\tilde{K}_{IC}^b + K_{IC}^s)^2 b^2 h}{\sqrt{\left(\sqrt{\pi} - \frac{a}{h} - 1\right) \cdot \left(\frac{2\pi - 1}{2\pi}\right)}} \frac{8}{l^2},$$

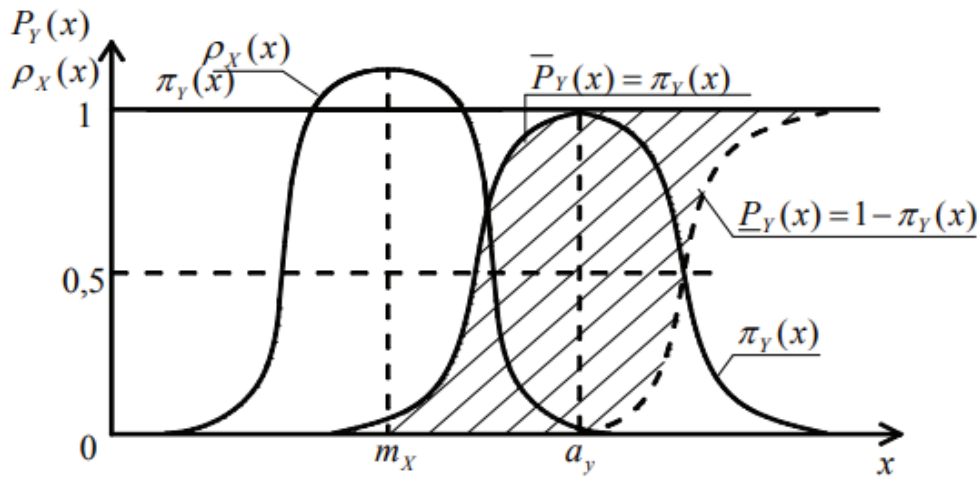


Figure 5. Probability density function of random variable X and distribution function of fuzzy variable Y graphs.

where the wavy line denotes random variables.

$$\text{Let } \tilde{q} = X, \sqrt{\frac{l_{crc,ult} \cdot (\tilde{K}_{IC}^b + K_{IC}^s)^2 b^2 h}{\left(\sqrt{\pi} - \frac{a}{h} - 1\right) \cdot \left(\frac{2\pi - 1}{2\pi}\right)}} \frac{8}{l^2} = Y.$$

$$m_y = \sqrt{\frac{l_{crc,ult} \cdot \left(m_{K_{IC}^b} + K_{IC}^s\right)^2 b^2 h}{\left(\sqrt{\pi} - \frac{a}{h} - 1\right) \cdot \left(\frac{2\pi - 1}{2\pi}\right)}} \frac{8}{l^2},$$

Describing the load on the beam \tilde{q} and the critical stress intensity coefficient \tilde{K}_{IC}^b by the normal distribution, the probability of non-failure can be found as:

$$S_y = \sqrt{\frac{l_{crc,ult} \cdot \left(S_{K_{IC}^b}\right)^2 b^2 h}{\left(\sqrt{\pi} - \frac{a}{h} - 1\right) \cdot \left(\frac{2\pi - 1}{2\pi}\right)}} \frac{8}{l^2}.$$

$$P = \Pr(X \leq Y) = \Phi(\beta) = \Phi\left(\frac{m_y - m_x}{\sqrt{S_x^2 + S_y^2}}\right), \quad (5)$$

where m_x and m_y – expected values of X and Y ; S_x and S_y – standard deviations of X and Y ; $\Phi()$ – value of the Laplace integral function; β – reliability index [11-13].

The function parameters are calculated by the follows equations: $m_x = m_q$, $S_x = S_q$,

If the reliability (probability of non-failure) requirements are not met, the design load value is reduced to the new value $m_x = m_q$ and the reliability value is recalculated.

However, it is not always possible to get a large amount of statistical data about the parameter \tilde{K}_{IC}^b . An approach based on a combination of probability theory and fuzzy set theory can be used in this case. Fuzzy set theory allows to model the variability of a random (fuzzy) variable with a small amount of statistical data. Figure 5 shows the probability density function of random variable X and the distribution function of fuzzy variable Y graphs

The reliability interval can be calculated using the following equations:

$$\underline{P} = 1 - \int_0^{a_y} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x-m_x)^2}{2S_x^2}\right] \cdot \exp\left[-\left(\frac{x-a_y}{b_y}\right)^2\right] dx -$$

$$- \int_{a_y}^{\infty} \exp\left[-\frac{(x-m_x)^2}{2S_x^2}\right] dx;$$

$$\bar{P} = 1 - \int_0^{a_y} 0 dx -$$

$$- \int_{a_y}^{\infty} \exp\left[-\frac{(x-m_x)^2}{2S_x^2}\right] \cdot \left[1 - \exp\left[-\left(\frac{x-a_y}{b_y}\right)^2\right]\right] dx$$

where

$$a_y = \frac{Y_{\max} + Y_{\min}}{2}$$

– “mean” value;

$$b_y = \frac{Y_{\max} + Y_{\min}}{2 \cdot \sqrt{-\ln \alpha}}$$

– measure of variability; α - cut (risk) level [14-15]; Y_{\max} , Y_{\min} – minimum and maximum values based on test results and calculations. The reliability is described by the interval $[\underline{P}; \bar{P}]$.

For the above-described problem:

$$Y_{\max/\min} = \sqrt{\frac{l_{crc,ult} \cdot (K_{IC,max/\min}^b + K_{IC}^s)^2 b^2 h}{\left(\sqrt{\pi} - \frac{a}{h} - 1\right) \cdot \left(\frac{2\pi - 1}{2\pi}\right)}} \frac{8}{l^2}.$$

The optimal level of non-failure probability (or reliability index) should be set taking into account the acceptable risk [16-18].

4. CONCLUSIONS

1. The article presents probabilistic approach to evaluation the ultimate load on flexural reinforced concrete elements on crack length criterion based of fracture mechanics;
2. The functional relationship between the critical stress intensity coefficient of concrete and the design concrete resistance is given. It can be used in inspections and maintenances of RC elements;
3. An ultimate crack length should be set for reinforced concrete elements individually;
4. The article describes the reliability analysis method for reinforced concrete flexural elements on crack length with limited statistical data.

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