GLOBAL ASYMPTOTICS OF THE FILTRATION PROBLEM IN A POROUS MEDIUM

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Abstract: Filtration of the suspension in a porous medium is important when strengthening the soil and creating watertight partitions for the construction of tunnels and underground structures. A model of deep bed filtration with variable porosity and fractional flow, and a size-exclusion mechanism of particle retention are considered. A global asymptotic solution is constructed in the entire domain in which the filtering process takes place. The obtained asymptotics is close to the numerical solution.

Keywords: deep bed filtration, porous medium, suspended and retained particles, mathematical model, global asymptotics

ГЛОБАЛЬНАЯ АСИМПТОТИКА ЗАДАЧИ ФИЛЬТРАЦИИ В ПОРИСТОЙ СРЕДЕ

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Аннотация: Задачи фильтрации суспензии в пористой среде важны при укреплении грунта и создании водонепроницаемых перегородок для строительства туннелей и подземных сооружений. Рассматривается модель долговременной глубинной фильтрации с переменной пористостью и доступным потоком, и размерным механизмом задержания частиц. Строится глобальное асимптотическое решение во всей области, в которой происходит процесс фильтрации. Найденная асимптотика всюду близка к численному решению.

Ключевые слова: долговременная глубинная фильтрация, пористая среда, взвешенные и осажденные частицы, математическая модель, глобальная асимптотика

1. INTRODUCTION

Construction of tunnels and underground structures construction requires strengthening the soil and creation of watertight partitions protecting the structures from ground and flood waters. When pumping a liquefied solution of concrete into a porous soil, the grout penetrates into the pores and is filtered by porous rock. After drying and hardening of grout, the soil becomes durable and waterproof [1, 2].

When filtering the suspension in a porous medium, part of the solid particles is transferred from the inlet to the outlet of a porous sample, and some particles get stuck in the pores and form a

deposit. Particles retention is influenced by different forces depending on the chemical and physical composition of the porous medium and suspension. There are many different filtration models with various mechanisms for the transport and deposition of particles [3-5]. If the size distribution of particles and pores overlap, then the main reason for particle retention becomes size-exclusion: solid suspended particles freely pass through large pores and get stuck at the inlet of small pores [6].

The classical mathematical model of deep bed filtration includes the mass balance equation for suspended and retained particles and the kinetic equation for deposit growth [7]. With a relative-

ly small number of solid particles in suspension, deposit growth is proportional to the first degree of suspended particles concentration. In the general case the proportionality coefficient between the growth rate of the deposit and the suspended concentration depends nonlinearly on the retained particles concentration and is called the filtration coefficient. The exact solution of this problem was obtained in [8].

More complex filtration models suggest that the porosity and the fractional flow are not constant, but depend on retained particles concentration [9]. For a complicated system of equations, local asymptotic solutions were obtained near the porous medium inlet, on the concentrations front, and for large time [10–12]. However, it is not possible to construct a global asymptotics on the basis of these solutions, since the applicability domains of local asymptotics do not cover the entire zone in which the system of equations is considered.

In this paper, a new method for constructing a global asymptotic solution has been developed. On its basis, the global asymptotics of the deep bed filtration problem of a monodisperse suspension in a homogeneous porous medium is obtained. Coordinate transformation leads to a variable that is small in the entire filtration zone. The global asymptotic solution is constructed as a series in powers of the new variable. The asymptotics is compared with the numerical solution obtained by the finite difference method [13–15]. The calculations show that the asymptotic and numerical solutions are close in the entire filtration zone.

2. MATHEMATICAL MODEL

A one-dimensional model of deep bed filtration of suspensions and colloids in a porous medium with variable porosity and fractional flow in the domain $\Omega = \{0 \leq x \leq 1, t \geq 0\}$ is determined by a quasilinear hyperbolic system of two first-order equations

$$\frac{\partial}{\partial t}(g(S)C) + \frac{\partial}{\partial x}(f(S)C) + \frac{\partial S}{\partial t} = 0; \quad (1)$$

$$\frac{\partial S}{\partial t} = \Lambda(S)C. \quad (2)$$

Here $g(S)$, $f(S)$ are smooth positive functions for $S \geq 0$; the blocking filtration coefficient $\Lambda(S)$ is positive for $0 \leq S < S_{\max}$ and turns to zero for $S = S_{\max}$; unknown functions $C(x, t)$; $S(x, t)$ are concentrations of suspended and retained particles.

The unique solution to the system (1), (2) is determined by the initial and boundary conditions

$$C|_{x=0} = 1; \quad (3)$$

$$C|_{t=0} = 0; \quad S|_{t=0} = 0. \quad (4)$$

Condition (3) means that a suspension of constant concentration is injected at the inlet of the porous medium; condition (4) corresponds to the absence of suspended and retained particles in the porous medium at the initial moment.

The concentrations front of suspended and retained particles moves in a porous medium from the inlet to the outlet at a constant velocity $v = f(0)/g(0)$ and is determined by formula $t = \alpha x$, $\alpha = g(0)/f(0)$.

Before the front in the domain $\Omega_0 = \{0 \leq x \leq 1, 0 \leq t \leq \alpha x\}$, the porous medium is empty, and the problem has a zero solution. Behind the concentration front in the domain $\Omega_s = \{0 \leq x \leq 1, t \geq \alpha x\}$, the porous medium contains suspended and retained particles, and the solution is positive. Since conditions (3) and (4) are not consistent at the origin, the solution $C(x, t)$ is discontinuous on the concentration front; the solution $S(x, t)$ is continuous in the domain Ω and vanishes on the front

$$S|_{t=\alpha x} = 0. \quad (5)$$

In the domain Ω_s , the solution of the original problem (1) - (4) coincides with the solution of problem (1) - (3), (5).

For constant coefficients $g = g_0$, $f = f_0$ of equation (1) and linear blocking filtration coefficient (Langmuir coefficient)

$$\Lambda(S) = a - bS; \quad a > 0, b > 0, \quad (6)$$

the simplified problem (1) - (3), (5) in the domain Ω_s has an exact solution

$$\begin{aligned} C(x, t) &= \frac{e^{b(t-\alpha x)}}{e^{b(t-\alpha x)} + e^{ax/f_0} - 1}; \\ S(x, t) &= \frac{a}{b} \cdot \frac{e^{b(t-\alpha x)} - 1}{e^{b(t-\alpha x)} + e^{ax/f_0} - 1}. \end{aligned} \quad (7)$$

In the general case, the exact solution of problem (1) - (3), (5) cannot be written down by explicit formulas [16]; therefore, asymptotics are constructed.

In characteristic variables $\tau = t - \alpha x$, $y = x$, the concentration front is given by the equation $\tau = 0$. In the half-strip $\Omega' = \{0 \leq y \leq 1, \tau \geq 0\}$, problem (1) - (3), (5). takes the form

$$\frac{\partial((g(S) - \alpha f(S))C)}{\partial \tau} + \frac{\partial(f(S)C)}{\partial y} + \frac{\partial S}{\partial \tau} = 0; \quad (8)$$

$$\frac{\partial S}{\partial \tau} = \Lambda(S)C; \quad (9)$$

$$C|_{y=0} = 1; \quad S|_{\tau=0} = 0. \quad (10)$$

$$\begin{aligned} &b(1-T)\varepsilon S(G(S) - \alpha F(S))\frac{\partial C}{\partial T} + a(1-z)\left(1 + \varepsilon \frac{SF(S)}{f_0}\right)\frac{\partial C}{\partial z} + \\ &+ \left(\varepsilon(SG(S) - \alpha SF(S))'\Lambda(S)C + \varepsilon(SF(S))'\frac{a}{f_0}(1-z)\frac{\partial S}{\partial z} + \Lambda(S)\right)C = 0; \end{aligned} \quad (14)$$

$$b(1-T)\frac{\partial S}{\partial T} = (a - bS + \varepsilon S^2\lambda(S))C \quad (15)$$

$$C|_{z=0} = 1; \quad S|_{T=0} = 0 \quad (16)$$

A solution of equations (8)-(10) is obtained in the form of a double series in powers of ε , z :

Near the concentration front, the local asymptotic solution of the problem (8) - (10) is constructed using the small parameter τ .

3. ASYMPTOTIC SOLUTION

A change of variables in the system (8) - (10)

$$z = 1 - e^{-ay/f_0}; \quad T = 1 - e^{-b\tau} \quad (11)$$

allows to use z as a small parameter, because for $(y, \tau) \in \Omega'$ the variables z , T belong to the domain $\Omega'' = \{0 \leq z \leq 1 - e^{af_0} < 1, 0 \leq T < 1\}$.

In the new variables, the solution (7) of the simplified problem takes the form

$$C(z, \tau) = \frac{1-z}{1-zT}; \quad S(z, \tau) = \frac{a}{b} \cdot \frac{(1-z)T}{1-zT}. \quad (12)$$

Assume that the coefficients of equation (8) are almost constant, and the filtration coefficient differs little from a linear function:

$$\begin{aligned} g(S) &= g_0 + \varepsilon SG(S); \\ f(S) &= f_0 + \varepsilon SF(S); \\ \Lambda(S) &= a - bS + \varepsilon S^2\lambda(S) \end{aligned} \quad (13)$$

Here $\varepsilon > 0$ is a small parameter.

In the domain Ω'' the problem (8)-(10) takes the form

$$C(\varepsilon, z, T) = 1 + \sum_{i=0}^n \sum_{j=1}^m c_{i,j}(T) \varepsilon^i z^j / j! = 1 + z(c_{0,1} + \varepsilon c_{1,1} + \varepsilon^2 c_{2,1} + \dots) + \frac{z^2}{2}(c_{0,2} + \varepsilon c_{1,2} + \varepsilon^2 c_{2,2} + \dots) + \dots; \quad (17)$$

$$S(\varepsilon, z, T) = \sum_{i=0}^n \sum_{j=0}^m s_{i,j}(T) \varepsilon^i z^j / j! = (s_{0,0} + \varepsilon s_{1,0} + \varepsilon^2 s_{2,0} + \dots) + z(s_{0,1} + \varepsilon s_{1,1} + \varepsilon^2 s_{2,1} + \dots) + \frac{z^2}{2}(s_{0,2} + \varepsilon s_{1,2} + \varepsilon^2 s_{2,2} + \dots) + \dots. \quad (18)$$

The coefficients $c_{i,j}$, $s_{i,j}$ depend only on T .

From conditions (16): $c_{i,0} = 0$; $s_{ij}|_{T=0} = 0$.

Substituting the expansions (17), (18) into equations (14), (15) and equating the terms with the same degrees of ε , z , implies a recurrent system of differential and algebraic equations

$$\varepsilon^0 z^0 : b(1-T)s'_{0,0} = (a - bs_{0,0}) \cdot 1;$$

$$\varepsilon^0 z^0 : ac_{0,1} + (a - bs_{0,0}) \cdot 1 = 0;$$

$$\varepsilon^0 z^1 : b(1-T)s'_{0,1} = -bs_{0,1} \cdot 1 + (a - bs_{0,0})c_{0,1};$$

$$\varepsilon^0 z^1 : ac_{0,2} - ac_{0,1} - bs_{0,1} \cdot 1 + (a - bs_{0,0})c_{0,1} = 0;$$

$$\varepsilon^1 z^0 : b(1-T)s'_{1,0} = (-bs_{1,0} + s_{0,0}^2 \lambda(s_{0,0})) \cdot 1 + (a - bs_{0,0}) \cdot 0;$$

$$\varepsilon^1 z^0 : ac_{1,1} + a \frac{s_{0,0} F(s_{0,0})}{f_0} c_{0,1} + (SG(S) - \alpha SF(S))' \Big|_{S=s_{0,0}} \cdot (a - bs_{0,0}) \cdot 1 + (SF(S))' \Big|_{S=s_{0,0}} \frac{a}{f_0} s_{0,1} + (-bs_{1,0} + s_{0,0}^2 \lambda(s_{0,0})) \cdot 1 + (a - bs_{0,0}) \cdot 0 = 0;$$

$$\varepsilon^1 z^1 : b(1-T)s'_{1,1} = -bs_{1,1} + (S^2 \lambda(S))' \Big|_{S=s_{0,0}} \cdot s_{0,1} - bs_{1,0} c_{0,1} + (a - bs_{0,0}) c_{1,1} + s_{0,0}^2 \lambda(s_{0,0}) c_{0,1}.$$

Solving equations sequentially, the coefficients of expansions (17), (18) are obtained

$$s_{0,0} = \frac{a}{b} T; \quad c_{0,1} = T - 1; \quad s_{0,1} = \frac{a}{b} T(T - 1);$$

$$c_{0,2} = 2T(T - 1); \quad \dots$$

Since the parameters ε , z are small in the entire domain Ω'' , the expansions (17), (18) determine the global asymptotic solution of problem (14) - (16) in the domain Ω'' .

4. NUMERICAL CALCULATION

The calculations were performed for the coefficients calculated in the laboratory of the University of Adelaide (Australia) in the study of deep bed filtration of a monodisperse suspension with particles of radius $r = 2.179 \mu\text{m}$ in a homogeneous porous medium [10]

$$\begin{aligned} g(S) &= 0.9743 - 8.88 \cdot 10^{-14} S + 1.27 \cdot 10^{-11} S^2 - 1.24 \cdot 10^{-9} S^3; \\ f(S) &= 0.9947 + 6.27 \cdot 10^{-5} S - 2.9 \cdot 10^{-8} S^2 + 6.21 \cdot 10^{-10} S^3; \\ \Lambda(S) &= 0.51 - 5.956 \cdot 10^{-3} S + 2.29 \cdot 10^{-6} S^2 + 1.35 \cdot 10^{-8} S^3. \end{aligned} \quad (19)$$

In expansions (17), (18) of the nonlinear problem (14) - (16), terms were calculated up to ε^1 and z^5 . The asymptotics was compared with a numerical calculation performed by the finite difference method. The equations (1), (2) was approximated by an original second-order scheme using the trapezium method [17]. The obtained solution was compared with the exact

solution (7) of the simplified problem with a linear filtration coefficient.

Figures 1-5 show graphs of global asymptotics (red line), numerical solution (blue line), and exact solution (7) (green line). Full graph images are in the center of Fig. 1–4; their enlarged fragments are shown on the left and right.

Figure 1. Graphs of suspended particles concentration $C(1,t)$ at the porous medium outlet.

Figure 2. Graphs of retained particles concentration $S(1,t)$ at the porous medium outlet.

Figure 3. Graphs of suspended particles concentration $C(x,100)$ at the fixed moment $t=100$.

Figure 4. Graphs of retained particles concentration $S(x,100)$ at the fixed moment $t=100$.

a) $C(x, 800)$ b) $S(x, 800)$

Figure 5. Graphs of suspended and retained particles concentration at the fixed moment $t = 800$.

Graphs of global asymptotics and numerical solutions are close to each other, the discrepancy increases with increasing coordinate x . The greatest deviation is observed at the outlet $x=1$ at time $t = 200$: 1,6% for C and 1,1% for S . The graphs of the exact solution (7) for the linear case are further away from the numerical solution than the asymptotic graphs.

5. CONCLUSION

A mathematical model of deep bed filtration with variable porosity and fractional flow does not have an exact solution in explicit form. Asymptotic and numerical methods are used to obtain the solution.

The global asymptotics approximates well the solution in the entire domain in which the suspension or colloid is filtered. Deviation increases with time.

The exact solution (7) of the simplified problem is not suitable for approximating the solution of a complex system.

The global asymptotics extends the possibilities of solving the inverse filtration problem — determining the coefficients (19) of system (1) - (4) from a known suspended particles concentration at the porous medium outlet [18–20].

The obtained asymptotic solutions can be used to fine-tune laboratory experiments [21], which significantly reduces the amount and cost of research.

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