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ON HAMILTONIAN FORMULATIONS AND CONSERVATION LAWS FOR PLATE THEORIES OF VEKUA-AMOSOV TYPE

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Abstract: Some variants of the generalized Hamiltonian formulation of the plate theory of I. N. Vekua – A. A. Amosov type are presented. The infinite dimensional formulation with one evolution variable, or an "instantaneous" formalism, as well as the de Donder – Weyl one are considered, and their application to the numerical simulation of shell and plate dynamics is briefly discussed. The main conservation laws are formulated for the general plate theory of N^{th} order, and the possible motion integrals are introduced.

Keywords: refined shell theory, analytical mechanics of continua, Hamiltonian formalism, de Donder – Weyl formulation, conservation laws

О ГАМИЛЬТОНОВЫХ ФОРМУЛИРОВКАХ И ЗАКОНАХ СОХРАНЕНИЯ В ТЕОРИЯХ ПЛАСТИН ТИПА И.Н. ВЕКУА – А.А. АМОСОВА

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Резюме: Предложены различные варианты обобщенных гамильтоновых формулировок общей теории оболочек типа И.Н. Векуа – А.А. Амосова. Рассмотрены формулировки на базе бесконечномерного фазового пространства с одной эволюционной переменной, а также на основе формализма де Дондера – Вейля, обсуждаются возможности их приложения к численным решениям задач механики тонкостенных систем. Получены основные законы сохранения для теории оболочек *N*-го порядка и определены основные первые интегралы системы.

Ключевые слова: теория оболочек высшего порядка, аналитическая механика континуальных систем, формализм Гамильтона, формулировка де Дондера – Вейля, законы сохранения

1. INTRODUCTION

The modern theory of shells and plates deals with many different problems in thin-walled structures' mechanics [1]. One can note that "…new, more reliable 2D multi-field models are needed… for high-frequency vibrations of shells" [2] (see also [3-7]), in particular for functionally graded structures [8-10], as well as for wave propagation problems [11-13] and in the transient plate and shell dynamics [14]. A lot of different refined models for plates and shells exist, for instance the ones based on power series extension [4, 5, 8], or using the asymptotic integration [11-13, 15, 16]. An efficient approach that allows one to construct the hierarchy of various order shell and plate models was constructed by I.N. Vekua [17] on the basis of orthogonal series and further improved in [18-20]; this type of shell theories has shown its efficiency in dynamics, especially for transient problems [21].

The new variant of the shell theory of I. N. Vekua type obtained by A.A. Amosov [22, 23] is based on the use of the tensor algebra; it offers the improved formalization level close to the one of the finite element method [23, 24]. The further improvement of the Vekua – Amosov higher order theory of shells and plates consists in the use of Lagrangian formalism of

analytical mechanics extended to continuum systems [24-27].
The Lagrangian formulation of refined plate models is quite efficient in steady dynamics problems such as [28, 29]. At the same time it is known that the Hamiltonian approach [30] provides some advantages (e. g. see [31]). Some attempts to use the features of symplecticity of Hamiltonian structures in statics were made in [32-35] where the so-called "instantaneous" formalism was applied and the Hamiltonian systems with operator coefficients were studied (an approach similar to [36-38]). This way leads to first-order "evolution" differential equations in time domain with second-order spatial derivatives. An alternative approach proposed by N.A. Kilchevskiy [39] deals with the Legendre transform considering both time and space derivatives; it leads to the finite dimensional phase space formalism well known in the general field theory as a "polymomentum formalism" [40, 41]. Various types of Lepagean equivalents may be used to obtain different Hamiltonian systems while the simplest one corresponds to the de Donder – Weyl formalism [40]. Both "instantanelus" and de Donder formulations for the plate theory of Nth order are constructed below, and main conservation laws are considered.

2. FUNDAMENTALS OF THE PLATE THEORY OF NTH ORDER

Let the plate be a three-dimensional body bounded by face planes $S_\pm$ and a lateral surface S_B [23-27, 47], with the mid-plane S and the plate thickness denoted as $2h$ [25].

$$V \subset \mathbb{R}^3, \bar{V} = V \cup \partial V, \partial V = S_B \oplus S_\pm;$$
$$\forall M_* \in \bar{V}, M \in S \quad \mathbf{R}(M_*) = \mathbf{r}(M) + h\zeta\mathbf{n}.$$

The two-dimensional model of a plate consists in the two-dimensional manifold S,

$$\bar{S} = S \cup (\partial S = S \cap S_B)$$

,

with the curvilinear chart $\xi^\alpha \in D_\xi \subseteq \mathbb{R}^2$, $\alpha = 1,2$, so that $\forall M \in \bar{S}$ $\mathbf{R}(M) \equiv \mathbf{r}(\xi^\alpha)$. Two base vectors

$$\mathbf{r}_\alpha = \partial_\alpha \mathbf{r}, \ \partial_\alpha \equiv \partial / \partial \xi^\alpha$$

corresponding to ξ^α induce the metrics

$$a_{\alpha\beta} = \mathbf{r}_\alpha \cdot \mathbf{r}_\beta;$$

the normal unit is defined as

$$\mathbf{n} = a^{-1/2}(\mathbf{r}_1 \times \mathbf{r}_2), \ a = \det a_{\alpha\beta}.$$

Let

$$\mathbf{u} = u_\alpha \mathbf{r}^\alpha + u_3 \mathbf{n}$$

be the spatial displacement vector field. Thus, the three-dimensional problem statement of elasticity theory is based on the Hamilton principle [25, 26]:

$$\delta \int_{t_0}^{t_1} \left(\int_V \mathcal{L}_V dV + \int_{\partial V} \mathcal{L}_{\partial V} dS \right) dt = 0. \qquad (2.1)$$

Let us consider the Lagrangian continuum system defined within the configuration manifold Ω_N with the field variables of the 1st kind $u_\alpha^{(k)}$, $u_3^{(k)}$ being the expansion factors of the spatial displacement field $\mathbf{u}$ with respect to the biorthogonal function system $p_{(k)}(\zeta)$, $p^{(k)}(\zeta)$ [26, 27], so that we have

$$\mathbf{u}(t, \xi^\alpha, \zeta) \approx \left(u_\alpha^{(k)} \mathbf{r}^\alpha + u_3^{(k)} \mathbf{n} \right) p_{(k)}(\zeta), \ k = \overline{0, N}.$$

Here the components $u_\alpha(t, \xi^\alpha, \zeta)$, $u_3(t, \xi^\alpha, \zeta)$ are assumed to be square integrable over

$[-1,1]\ni\zeta$ [25-27]. In general, the convergence is supposed folloving [15].
The Lagrangian densitiy can be now defined on S and ∂S as follows [33, 35]:

$$
\begin{aligned}
&\mathcal{L}_S\left(u_\alpha^{(k)}, u_3^{(k)}, \dot{u}_\alpha^{(k)}, \dot{u}_3^{(k)}, \bar{\nabla}_\beta u_\alpha^{(k)}, \bar{\nabla}_\beta u_3^{(k)}\right)= \\
&=\tfrac{1}{2}\rho_{(k)}^{(m)}\left(\dot{u}_{(m)}^{\alpha}\dot{u}_\alpha^{(k)}+\dot{u}_{(m)}^{3}\dot{u}_3^{(k)}\right)+F_{(k)}^{i}u_i^{(k)}- \\
&-\tfrac{1}{2}\left(C_{(km)}^{\alpha\beta\gamma\delta}\bar{\nabla}_\gamma u_\delta^{(m)}+C_{(km)}^{\alpha\beta 3}u_3^{(m)}\right)\bar{\nabla}_\beta u_\alpha^{(k)}- \\
&-\tfrac{1}{2}\left(C_{(km)}^{3\beta\gamma 3}\bar{\nabla}_\gamma u_3^{(m)}+C_{(km)}^{3\beta\gamma}u_\gamma^{(m)}\right)\bar{\nabla}_\beta u_3^{(k)}- \qquad (2.2)\\
&-\tfrac{1}{2}h^{-1}\left(C_{(km)}^{33\gamma\delta}\bar{\nabla}_\gamma u_\delta^{(m)}+C_{(km)}^{333}u_3^{(m)}\right)D_{(n\cdot)}^{(\cdot k)}u_3^{(n)}- \\
&-\tfrac{1}{2}h^{-1}\left(C_{(km)}^{\alpha 3\gamma 3}\bar{\nabla}_\gamma u_3^{(m)}+C_{(km)}^{\alpha 3\gamma}u_\gamma^{(m)}\right)D_{(n\cdot)}^{(\cdot k)}u_\alpha^{(n)}; \\
&\mathcal{L}_{\partial S}\left(u_\alpha^{(k)}, u_3^{(k)}\right)=q_{B(k)}^{\alpha}u_\alpha^{(k)}+q_{B(k)}^{\alpha}u_\alpha^{(k)}.
\end{aligned}
$$

Here $\bar{\nabla}$ denotes the covariant derivative and $D_{(\cdot m)}^{(k\cdot)}$ are linear operators (see [23-29]):

$$
\begin{aligned}
&\rho_{(k)}^{(m)}=\left(\rho p^{(m)}, p_{(k)}\right)_1; \quad D_{(n\cdot)}^{(\cdot k)}=\left(dp_{(n)}/d\zeta, p^{(k)}\right)_1; \\
&C_{(km)}^{333}=h^{-1}D_{(k\cdot)}^{(\cdot n)}C_{(nm)}^{3333}; \quad C_{(km)}^{\alpha\beta 3}=h^{-1}D_{(k\cdot)}^{(\cdot n)}C_{(nm)}^{\alpha\beta 33}; \\
&C_{(km)}^{3\beta\gamma}=h^{-1}D_{(k\cdot)}^{(\cdot n)}C_{(nm)}^{3\beta 3\gamma}; \quad C_{(km)}^{\alpha\beta 3}=h^{-1}D_{(k\cdot)}^{(\cdot n)}C_{(nm)}^{\alpha 3\beta 3}.
\end{aligned}
$$

Thus, the plate is defined as a two-dimensional continuum system within the field variables $u_\alpha^{(k)}$, $u_3^{(k)}$ and the Lagrangian densities $\mathcal{L}_S$, $\mathcal{L}_{\partial S}$. Finally, the generalized Lagrange equations of the second kind [25]

$$
\begin{aligned}
&\partial_t\frac{\partial\mathcal{L}_S}{\partial\dot{u}_\alpha^{(k)}}+\bar{\nabla}_\beta\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_\alpha^{(k)}}-\frac{\partial\mathcal{L}_S}{\partial u_\alpha^{(k)}}=0; \\
&\partial_t\frac{\partial\mathcal{L}_S}{\partial\dot{u}_3^{(k)}}+\bar{\nabla}_\beta\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_3^{(k)}}-\frac{\partial\mathcal{L}_S}{\partial u_3^{(k)}}=0
\end{aligned}
\qquad (2.3)
$$

can be formulated in the particular case of the Nth order plate theory as (2.4) (e. g. see [25])

$$
\begin{aligned}
&\rho_{(k)}^{(m)}\ddot{u}_{(m)}^{\alpha}=\bar{\nabla}_\beta\sigma_{(k)}^{\alpha\beta}-h^{-1}D_{(k\cdot)}^{(\cdot m)}\sigma_{(m)}^{\alpha 3}+\bar{F}_{(k)}^{\alpha}; \\
&\rho_{(k)}^{(m)}\ddot{u}_{(m)}^{3}=\bar{\nabla}_\beta\sigma_{(k)}^{3\beta}-h^{-1}D_{(k\cdot)}^{(\cdot m)}\sigma_{(m)}^{33}+\bar{F}_{(k)}^{3};
\end{aligned}
\qquad (2.4)
$$

as well their natural boundary conditions (2.5)

$$
\begin{aligned}
&\left(\sigma_{(k)}^{\alpha\beta}\nu_\beta-q_{B(k)}^{\alpha}\right)\delta u_\alpha^{(k)}\Big|_{S_B}=0; \\
&\left(\sigma_{(k)}^{3\beta}\nu_\beta-q_{B(k)}^{3}\right)\delta u_3^{(k)}\Big|_{S_B}=0;
\end{aligned}
\qquad (2.5)
$$

while the corresponding initial conditions can be written as (2.6) (see [24-26]):

$$
\begin{aligned}
&u_\alpha^{(k)}\Big|_{t=t_0}=U_\alpha^{(k)}; \; u_3^{(k)}\Big|_{t=t_0}=U_3^{(k)}; \\
&\dot{u}_\alpha^{(k)}\Big|_{t=t_0}=V_\alpha^{(k)}; \; \dot{u}_3^{(k)}\Big|_{t=t_0}=V_3^{(k)}.
\end{aligned}
\qquad (2.6)
$$

Let us note that the generalized tangential and shear forces can be defined by the partial derivatives of the Lagrangian density $\mathcal{L}_S$:

$$
\sigma_{(k)}^{\alpha\beta}=-\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_\alpha^{(k)}}, \; \sigma_{(k)}^{3\beta}=-\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_3^{(k)}}. \qquad (2.7)
$$

Finally, the total Lagrangian for the plate can be determined as follows:

$$
L=\int_S\mathcal{L}_S dS \quad \int_{\partial S}\mathcal{L}_{\partial S}d\Gamma.
$$

3. ENERGY AND MOMENTUM CONSERVATION LAWS IN THE PLATE THEORY OF NTH ORDER

Let us define the total time derivative of the Lagrangian densities given by (2.2):

$$\frac{d}{dt}\mathcal{L}_S\left(u_\alpha^{(k)}, u_3^{(k)}, \dot{u}_\alpha^{(k)}, \dot{u}_3^{(k)}, \overline{\nabla}_\beta u_\alpha^{(k)}, \overline{\nabla}_\beta u_3^{(k)}\right)=$$
$$=\frac{\partial\mathcal{L}_S}{\partial u_\alpha^{(k)}}\dot{u}_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial u_3^{(k)}}\dot{u}_3^{(k)}+\frac{\partial\mathcal{L}_S}{\partial \dot{u}_\alpha^{(k)}}\ddot{u}_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial \dot{u}_3^{(k)}}\ddot{u}_3^{(k)}+$$
$$+\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_\beta u_\alpha^{(k)}}\overline{\nabla}_\beta\dot{u}_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_\beta u_3^{(k)}}\overline{\nabla}_\beta\dot{u}_3^{(k)};$$
$$\frac{d}{dt}\mathcal{L}_{\partial S}\left(u_\alpha^{(k)}, u_3^{(k)}\right)=\overline{q}_{(k)}^{\alpha}\dot{u}_\alpha^{(k)}+\overline{q}_{(k)}^{3}\dot{u}_3^{(k)}.$$

Accounting for the Gauss-Ostrogradsky theorem, we can represent the corresponding time derivative for the total Lagrangian as

$$\begin{aligned}&\int_S\frac{d\mathcal{L}_S}{dt}dS+\int_{\partial S}\frac{d\mathcal{L}_{\partial S}}{dt}d\Gamma=\\ &=\int_S\frac{d}{dt}\left(\frac{\partial\mathcal{L}_S}{\partial\dot{u}_\alpha^{(k)}}\dot{u}_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial\dot{u}_3^{(k)}}\dot{u}_3^{(k)}\right)dS+\\ &-\int_S\left[\partial_t\frac{\partial\mathcal{L}_S}{\partial\dot{u}_\alpha^{(k)}}+\overline{\nabla}_\beta\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_\beta u_\alpha^{(k)}}-\frac{\partial\mathcal{L}_S}{\partial u_\alpha^{(k)}}\right]\dot{u}_\alpha^{(k)}dS-\\ &-\int_S\left[\partial_t\frac{\partial\mathcal{L}_S}{\partial\dot{u}_3^{(k)}}+\overline{\nabla}_\beta\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_\beta u_3^{(k)}}-\frac{\partial\mathcal{L}_S}{\partial u_3^{(k)}}\right]\dot{u}_3^{(k)}dS+\\ &+\int_{\partial S}\left(\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_\beta u_\alpha^{(k)}}\dot{u}_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_\beta u_3^{(k)}}\dot{u}_3^{(k)}\right)\nu_\beta d\Gamma-\\ &+\int_{\partial S}\left(\frac{\partial\mathcal{L}_{\partial S}}{\partial u_\alpha^{(k)}}\dot{u}_\alpha^{(k)}+\frac{\partial\mathcal{L}_{\partial S}}{\partial u_3^{(k)}}\dot{u}_3^{(k)}\right)d\Gamma.\end{aligned}\tag{3.1}$$

Taking into account the generalized Lagrange equations of the second kind (2.3) for the plate theory of N[th] order [25-27] coinciding with the square-bracketed terms in (3.1) as well as their natural boundary conditions (2.5), we can write the total time derivative using the following representation:

$$\begin{aligned}\frac{dH^0}{dt}=&\int_{\partial S}\left(\sigma_{(k)}^{\alpha\beta}\nu_\beta-\overline{q}_{B(k)}^{\alpha}\right)\dot{u}_\alpha^{(k)}d\quad]\\ &+\int_{\partial S}\left(\sigma_{(k)}^{3\beta}\nu_\beta-\overline{q}_{B(k)}^{3}\right)\dot{u}_3^{(k)}d\Gamma,\end{aligned}\tag{3.2}$$

where the Hamiltonian is defined as follows:

$$H^0=\int_S\mathcal{H}^0 dS.$$

The surface density of the Hamiltonian $\mathcal{H}^0$ is constructed by means of the Legendre transform of the Lagrangian surface density $\mathcal{L}_S$ considering only time derivatives of the field variables $\dot{u}_\alpha^{(k)}$, $\dot{u}_3^{(k)}$ in terms of the so-called "infinite dimensional phase space", or "instantaneous" formulation [30, 42]:

$$\mathcal{H}^0=p_\alpha^{(k)}\dot{u}_{(k)}^{\alpha}+p_3^{(k)}\dot{u}_{(k)}^{3}-\mathcal{L}_S,\tag{3.3}$$

where the generalized momenta of the k-ке order are determined by the formulae

$$\begin{aligned}p_{(k)}^{\alpha}&=\frac{\partial\mathcal{L}_S}{\partial\dot{u}_\alpha^{(k)}}={}_{(k)}^{(m)}\dot{u}_{(m)}^{\alpha};\\ p_3^{(k)}&=\frac{\partial\mathcal{L}_S}{\partial\dot{u}_3^{(k)}}={}_{(k)}^{(m)}\dot{u}_{(m)}^{3}.\end{aligned}\tag{3.4}$$

The formula (3.3) yields that H^0 is the total energy; it follows from (2.2) and (3.4). It can be seen that the bracketed terms in (3.2) coincide with (2.5), therefore the total energy H^0 becomes the integral of motion for the plate if the boundary conditions on ∂S are exactly satisfied. Thus, the formula (3.2) defines the energy conservation. Moreover we can obtain the differential formulation of the conservation law for the energy density $\mathcal{H}^0$:

$$\dot{\mathcal{H}}^0=-\overline{\nabla}_\beta S^\beta;\tag{3.5}$$

here $\mathbf{S}=S^\beta\mathbf{r}_\beta$ is the tangent vector field of the energy flux [43]:

$$S^\beta=-s_{(k)}^{\alpha\beta}\dot{u}_\alpha^{(k)}\quad s_{(k)}^{3\beta}\dot{u}_3^{(k)}.\tag{3.6}$$

This quantity can be useful, in particular, for waveguide problems such as [28, 29].

Now let us consider the total derivatives of the Lagrangian densities with respect to the surface coordinates ξ^γ:

$$\frac{d}{d\xi^\gamma}\mathcal{L}_S\left(u_\alpha^{(k)},u_3^{(k)},\dot{u}_\alpha^{(k)},\dot{u}_3^{(k)},\bar{\nabla}_\beta u_\alpha^{(k)},\bar{\nabla}_\beta u_3^{(k)}\right)=$$

$$=\frac{\partial\mathcal{L}_S}{\partial u_\alpha^{(k)}}\bar{\nabla}_\gamma u_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial u_3^{(k)}}\bar{\nabla}_\gamma u_3^{(k)}+$$

$$+\frac{\partial\mathcal{L}_S}{\partial\dot{u}_\alpha^{(k)}}\bar{\nabla}_\gamma\dot{u}_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial u_3^{(k)}}\bar{\nabla}_\gamma\dot{u}_3^{(k)}+$$

$$+\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_\alpha^{(k)}}\bar{\nabla}_\gamma\bar{\nabla}_\beta\dot{u}_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_3^{(k)}}\bar{\nabla}_\gamma\bar{\nabla}_\beta\dot{u}_3^{(k)};$$

$$\frac{d\mathcal{L}_{\partial S}}{d\xi^\gamma}=\bar{q}_{(k)}^{\alpha}\bar{\nabla}_\gamma u_\alpha^{(k)}+\bar{q}_{(k)}^{3}\bar{\nabla}_\gamma u_3^{(k)}.$$

Keeping in mind that the Riemann-Christoffel tensor of the plane S vanishes,

$$R_{\alpha\beta i}^{\cdots j}=0,$$

and for the Gauss-Ostrogradsky theorem, we have

$$\int_S\frac{d\mathcal{L}_S}{d\xi^\gamma}dS+\int_{\partial S}\frac{d\mathcal{L}_{\partial S}}{d\xi^\gamma}d\Gamma=$$

$$=\int_S\frac{d}{dt}\left(\frac{\partial\mathcal{L}_S}{\partial\dot{u}_\alpha^{(k)}}\bar{\nabla}_\gamma u_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial\dot{u}_3^{(k)}}\bar{\nabla}_\gamma u_3^{(k)}\right)dS-$$

$$-\int_S\left[\partial_t\frac{\partial\mathcal{L}_S}{\partial\dot{u}_\alpha^{(k)}}+\bar{\nabla}_\beta\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_\alpha^{(k)}}-\frac{\partial\mathcal{L}_S}{\partial u_\alpha^{(k)}}\right]\bar{\nabla}_\gamma u_\alpha^{(k)}dS-$$

$$-\int_S\left[\partial_t\frac{\partial\mathcal{L}_S}{\partial\dot{u}_3^{(k)}}+\bar{\nabla}_\beta\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_3^{(k)}}-\frac{\partial\mathcal{L}_S}{\partial u_3^{(k)}}\right]\bar{\nabla}_\gamma u_\alpha^{(k)}dS+$$

$$+\int_{\partial S}\left(\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_\alpha^{(k)}}\bar{\nabla}_\gamma u_\alpha^{(k)}+\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_3^{(k)}}\bar{\nabla}_\gamma u_3^{(k)}\right)\nu_\beta d\Gamma+$$

$$+\int_{\partial S}\left(\frac{\partial\mathcal{L}_{\partial S}}{\partial u_\alpha^{(k)}}\bar{\nabla}_\gamma u_\alpha^{(k)}+\frac{\partial\mathcal{L}_{\partial S}}{\partial u_3^{(k)}}\bar{\nabla}_\gamma u_3^{(k)}\right)d\Gamma.$$

Accounting hence for the Lagrange equations (2.3) for plates [25-27], we have finally

$$\begin{aligned}\frac{dG_\gamma}{dt}&=\int_S\left(\sigma_{(k)}^{\alpha\beta}\nu_\beta-\bar{q}_{B(k)}^{\alpha}\right)\bar{\nabla}_\gamma u_\alpha^{(k)}d\Gamma+\\&+\int_S\left(\sigma_{(k)}^{3\beta}\nu_\beta-\bar{q}_{B(k)}^{3}\right)\bar{\nabla}_\gamma u_3^{(k)}d\Gamma,\end{aligned}\tag{3.7}$$

$$G_\gamma=\int_S\mathcal{G}_\gamma dS,\tag{3.8}$$

where the covariant components of the wave momentum vector [30] are defined by (3.9):

$$\mathcal{G}_\gamma=\frac{\partial\mathcal{L}_S}{\partial\dot{u}_\alpha^{(k)}}\bar{\nabla}_\gamma u_\alpha^{(k)}\cdot\frac{\partial\mathcal{L}_S}{\partial\dot{u}_3^{(k)}}\bar{\nabla}_\gamma u_3^{(k)}.\tag{3.9}$$

Thus, the covariant components G_γ of the total wave momentum vector (3.8) become the integrals of motion if the boundary conditions on ∂S coinciding with the bracketed terms in the formula (3.7) are satisfied exactly.

The differential form of the field momentum conservation law can be written as follows:

$$\dot{\mathcal{G}}_\gamma=\bar{\nabla}_\beta\mathcal{H}_{\cdot\gamma}^{\beta\cdot},\tag{3.10}$$

where the components of the Hamilton tensor are introduced by (3.11) accordingly to [40]:

$$\begin{aligned}\mathcal{H}_{\cdot\gamma}^{\beta\cdot}&=\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_\alpha^{(k)}}\bar{\nabla}_\gamma u_\alpha^{(k)}+\\&+\frac{\partial\mathcal{L}_S}{\partial\bar{\nabla}_\beta u_3^{(k)}}\bar{\nabla}_\gamma u_3^{(k)}-\delta_\gamma^\beta\mathcal{L}_S.\end{aligned}\tag{3.11}$$

Finally, the conservation law for the moments of wave momentum [30] can be formulated. Let us introduce the momentum moment field normal to the mid-plane S of the plate:

$$M_3=\int_S\mathcal{M}_3 dS,\quad\mathcal{M}_3=\epsilon_{\gamma\delta}{}^{\gamma}\mathcal{G}^\delta.\tag{3.12}$$

Similarly to the previously derived integral conservation laws, we obtain for the wave momentum moment (3.12) the following one:

$$\frac{dM_3}{dt} = -\int_S \epsilon_{\beta\gamma}\mathcal{H}^{\beta\gamma} dS$$
$$+\int_{\partial S} \epsilon_{\delta\gamma}\xi^{\delta} a^{\gamma\varepsilon}\left(\sigma^{\alpha\beta}_{(k)}\nu_{\beta} - \overline{q}^{\alpha}_{B(k)}\right)\overline{\nabla}_{\varepsilon}u^{(k)}_{\alpha} d\Gamma + \quad (3.13)$$
$$+\int_{\partial S} \epsilon_{\delta\gamma}\xi^{\delta} a^{\gamma\varepsilon}\left(\sigma^{3\beta}_{(k)}\nu_{\beta} - \overline{q}^{3}_{B(k)}\right)\overline{\nabla}_{\varepsilon}u^{(k)}_{3} d\Gamma.$$

The stress symmetry,

$$\sigma^{\alpha\beta} = \sigma^{\beta\alpha},$$

and the first term in (3.13) imply that the Hamilton tensor (3.11) is symmetric; the total moment of momentum (3.12) becomes integral of motion if the boundary conditions on the plate contour ∂S are satisfied exactly.

4. HAMILTON EQUATIONS OF THE NTH ORDER PLATE THEORY IN TERMS OF THE INSTANTANEOUS" HAMILTON FORMALISM

These equations can be obtained following J.B. Leech [30]. Let us consider the total differential of the Hamiltonian $\mathcal{H}^0$:

$$d\mathcal{H}^0 = \frac{\partial\mathcal{H}^0}{\partial u^{(k)}_{\alpha}} du^{(k)}_{\alpha} + \frac{\partial\mathcal{H}^0}{\partial u^{(k)}_{3}} du^{(k)}_{3} +$$
$$+\frac{\partial\mathcal{H}^0}{\partial p^{\alpha}_{(k)}} dp^{\alpha}_{(k)} + \frac{\partial\mathcal{H}^0}{\partial\overline{\nabla}_{\beta}u^{(k)}_{\alpha}} d\left(\overline{\nabla}_{\beta}u^{(k)}_{\alpha}\right) +$$
$$+\frac{\partial\mathcal{H}^0}{\partial p^{3}_{(k)}} dp^{3}_{(k)} + \frac{\partial\mathcal{H}^0}{\partial\overline{\nabla}_{\beta}u^{(k)}_{3}} d\left(\overline{\nabla}_{\beta}u^{(k)}_{3}\right).$$

On the other hand, from the definition for the $\mathcal{H}^0$ (3.3) we have accordingly to J. Leech [30]

$$d\mathcal{H}^0 = \quad \frac{\partial\mathcal{L}_S}{\partial u^{(k)}_{\alpha}} du^{(k)}_{\alpha} \quad \frac{\partial\mathcal{L}_S}{\partial u^{(k)}_{3}} du^{(k)}_{3}$$
$$-\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_{\beta}u^{(k)}_{\alpha}} d\left(\overline{\nabla}_{\beta}u^{(k)}_{\alpha}\right) - \frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_{\beta}u^{(k)}_{3}} d\left(\overline{\nabla}_{\beta}u^{(k)}_{3}\right) -$$
$$+p^{\alpha}_{(k)}\dot{u}^{(k)}_{\alpha} + p^{3}_{(k)}\dot{u}^{(k)}_{3} + \dot{u}^{(k)}_{\alpha}p^{\alpha}_{(k)} + \dot{u}^{(k)}_{3}p^{3}_{(k)},$$

and the following relations can be derived:

$$\frac{\partial\mathcal{H}^0}{\partial u^{(k)}_{\alpha}} = -\frac{\partial\mathcal{L}_S}{\partial u^{(k)}_{\alpha}}, \quad \frac{\partial\mathcal{H}^0}{\partial u^{(k)}_{3}} = -\frac{\partial\mathcal{L}_S}{\partial u^{(k)}_{3}}; \qquad (4.1)$$

$$\frac{\partial\mathcal{H}^0}{\partial\overline{\nabla}_{\beta}u^{(k)}_{\alpha}} = -\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_{\beta}u^{(k)}_{\alpha}};$$
$$\frac{\partial\mathcal{H}^0}{\partial\overline{\nabla}_{\beta}u^{(k)}_{3}} = -\frac{\partial\mathcal{L}_S}{\partial\overline{\nabla}_{\beta}u^{(k)}_{3}} \qquad (4.2)$$

as well as two Hamiltonian equations:

$$\dot{u}^{(k)}_{\alpha} = \frac{\partial\mathcal{H}^0}{\partial p^{\alpha}_{(k)}}, \quad \dot{u}^{(k)}_{3} = \frac{\partial\mathcal{H}^0}{\partial p^{3}_{(k)}}. \qquad (4.3)$$

Now let us consider the generalized momenta (3.4). Accounting for the Lagrange equations [25] results in the following relationships:

$$\dot{p}^{\alpha}_{(k)} = \overline{\nabla}_{\beta}\frac{\partial\mathcal{H}^0}{\partial\overline{\nabla}_{\beta}u^{(k)}_{\alpha}} - \frac{\partial\mathcal{H}^0}{\partial u^{(k)}_{\alpha}};$$
$$\dot{p}^{3}_{(k)} = \overline{\nabla}_{\beta}\frac{\partial\mathcal{H}^0}{\partial\overline{\nabla}_{\beta}u^{(k)}_{3}} - \frac{\partial\mathcal{H}^0}{\partial u^{(k)}_{3}}. \qquad (4.4)$$

They are the dynamics equations [30, 40]:

$$\dot{p}^{\alpha}_{(k)} = \overline{\nabla}_{\beta}\left(C^{\alpha\beta\gamma\delta}_{(nm)}\overline{\nabla}_{\delta}u^{(m)}_{\gamma} + C^{\alpha\beta3}_{(nm)}u^{(m)}_{3}\right) -$$
$$-h^{-1}D^{(\square n)}_{(k\square)}\left(C^{\alpha33\delta}_{(nm)}\overline{\nabla}_{\delta}u^{(m)}_{3} + C^{\alpha3\gamma}_{(nm)}u^{(m)}_{\gamma}\right) + F^{\alpha}_{(k)}; \qquad (4.5)$$
$$\dot{p}^{3}_{(k)} = \overline{\nabla}_{\beta}\left(C^{3\beta3\delta}_{(nm)}\overline{\nabla}_{\delta}u^{(m)}_{3} + \overline{C}^{3\beta\gamma}_{(nm)}u^{(m)}_{\gamma}\right) +$$
$$-h^{-1}D^{(\square n)}_{(k\square)}\left(\overline{C}^{33\gamma\delta}_{(nm)}\overline{\nabla}_{\delta}u^{(m)}_{\gamma} + \overline{C}^{333}_{(nm)}u^{(m)}_{3}\right) + F^{3}_{(k)} \qquad (4.6)$$

similar to the Lagrange equations [25-27]; their right hand sides contain second-order covariant derivatives of the field variables $u_\alpha^{(k)}$, $u_3^{(k)}$. The natural boundary conditions can be rewritten in the following notation:

$$\left.\left[\nu_\beta \frac{\partial \mathcal{H}^0}{\partial \bar{\nabla}_\beta u_\alpha^{(k)}} + \frac{\partial \mathcal{H}_{\partial S}}{\partial u_\alpha^{(k)}}\right]\delta u_\alpha^{(k)}\right|_{\partial S} = 0; \qquad (4.7)$$
$$\left.\left[\nu_\beta \frac{\partial \mathcal{H}^0}{\partial \bar{\nabla}_\beta u_\alpha^{(k)}} + \frac{\partial \mathcal{H}_{\partial S}}{\partial u_\alpha^{(k)}}\right]\delta u_\alpha^{(k)}\right|_{\partial S} = 0$$

where formally

$$\mathcal{H}_{\partial S} = \mathcal{L}_{\partial S}.$$

Finally, we have the initial conditions

$$u_\alpha^{(k)}\Big|_{t=t_0} = U_\alpha^{(k)};\ u_3^{(k)}\Big|_{t=t_0} = U_3^{(k)};$$
$$p_{(k)}^\alpha\Big|_{t=t_0} = P_{(k)}^\alpha;\quad p_{(k)}^3\Big|_{t=t_0} = P_{(k)}^3. \qquad (4.8)$$

The obtained formulation (4.3), (4.5), (4.6), (4.7), and (4.8) corresponds to the so-called “instantaneous” Hamiltonian formulation for continuum systems that is based on the infinite-dimensional phase spaces [42].

5. HAMILTON – DE DONDER – WEYL EQUATIONS OF THE N-TH ORDER PLATE THEORY

The Hamilton – de Donder – Weyl approach deals with finite dimensional phase spaces [41, 42] and is based on the set of polymomenta induced by spatial derivatives of field variables of the 1st kind (2.7).

Let us construct the new Hamiltonian density depending on momenta and polymomenta using the Legendre transform considering both time and space derivatives of $\dot{u}_\alpha^{(k)}$, $\dot{u}_3^{(k)}$:

$$\mathcal{H}^{DW}\left(u_\alpha^{(k)}, u_3^{(k)}, p_{(k)}^\alpha, p_{(k)}^3, \sigma_{(k)}^{\alpha\beta}, \sigma_{(k)}^{3\beta}\right) =$$
$$= p_{(k)}^\alpha \dot{u}_\alpha^{(k)} + p_{(k)}^3 \dot{u}_3^{(k)} + \qquad (5.1)$$
$$+\sigma_{(k)}^{\alpha\beta}\bar{\nabla}_\beta u_\alpha^{(k)} + \sigma_{(k)}^{3\beta}\bar{\nabla}_\beta u_3^{(k)} - \mathcal{L}_S$$

and let us compute its total differential:

$$d\mathcal{H}^{DW} = \frac{\partial \mathcal{H}^{DW}}{\partial p_{(k)}^\alpha} dp_{(k)}^\alpha + \frac{\partial \mathcal{H}^{DW}}{\partial p_{(k)}^3} dp_{(k)}^3 +$$
$$+\frac{\partial \mathcal{H}^{DW}}{\partial \sigma_{(k)}^{\alpha\beta}} d\sigma_{(k)}^{\alpha\beta} + \frac{\partial \mathcal{H}^{DW}}{\partial \sigma_{(k)}^{3\beta}} d\sigma_{(k)}^{3\beta} +$$
$$+\frac{\partial \mathcal{H}^{DW}}{\partial u_\alpha^{(k)}} du_\alpha^{(k)} + \frac{\partial \mathcal{H}^{DW}}{\partial u_3^{(k)}} du_3^{(k)}.$$

On the other hand, from the definition given by Legendre transform (5.1) we obtain

$$d\mathcal{H}^{DW} = \dot{u}_\alpha^{(k)} dp_{(k)}^\alpha + \dot{u}_3^{(k)} dp_{(k)}^3 +$$
$$+\bar{\nabla}_\beta u_\alpha^{(k)} d\sigma_{(k)}^{\alpha\beta} + \bar{\nabla}_\beta u_3^{(k)} d\sigma_{(k)}^{3\beta} +$$
$$+p_{(k)}^\alpha d\dot{u}_\alpha^{(k)} + p_{(k)}^3 d\dot{u}_3^{(k)} +$$
$$+\sigma_{(k)}^{\alpha\beta} d\left(\bar{\nabla}_\beta u_\alpha^{(k)}\right) + \sigma_{(k)}^{3\beta} d\left(\bar{\nabla}_\beta u_3^{(k)}\right) - \qquad (5.2)$$
$$-\frac{\partial \mathcal{L}_S}{\partial \dot{u}_\alpha^{(k)}} d\dot{u}_\alpha^{(k)} - \frac{\partial \mathcal{L}_S}{\partial \dot{u}_3^{(k)}} d\dot{u}_3^{(k)} -$$
$$-\frac{\partial \mathcal{L}_S}{\partial \bar{\nabla}_\beta u_\alpha^{(k)}} d\bar{\nabla}_\beta u_\alpha^{(k)} - \frac{\partial \mathcal{L}_S}{\partial \bar{\nabla}_\beta u_3^{(k)}} d\bar{\nabla}_\beta u_3^{(k)} -$$
$$-\frac{\partial \mathcal{L}_S}{\partial u_\alpha^{(k)}} du_\alpha^{(k)} - \frac{\partial \mathcal{L}_S}{\partial u_3^{(k)}} du_3^{(k)},$$

therefore we have the following relations for the partial derivatives of the Lagrangian and Hamiltonian surface densities:

$$\frac{\partial \mathcal{H}^{DW}}{\partial u_\alpha^{(k)}} = -\frac{\partial \mathcal{L}_S}{\partial u_\alpha^{(k)}},\quad \frac{\partial \mathcal{H}^{DW}}{\partial u_3^{(k)}} = -\frac{\partial \mathcal{L}_S}{\partial u_3^{(k)}} \qquad (5.3)$$

as well as the first pair of the canonical Hamilton – de Donder – Weyl equations:

$$\dot{u}_\alpha^{(k)} = \frac{\partial \mathcal{H}^{DW}}{\partial p^\alpha_{(k)}}, \quad \dot{u}_3^{(k)} = \frac{\partial \mathcal{H}^{DW}}{\partial p^3_{(k)}} \tag{5.4}$$

and the second quasi-canonical pair:

$$\bar{\nabla}_\beta u_\alpha^{(k)} = \frac{\partial \mathcal{H}^{DW}}{\partial \sigma^{\alpha\beta}_{(k)}}, \quad \bar{\nabla}_\beta u_3^{(k)} = \frac{\partial \mathcal{H}^{DW}}{\partial \sigma^{3\beta}_{(k)}}. \tag{5.5}$$

The equations (5.4) define the generalized velocities, or field variables of the 2nd kind, whereas the equations (5.5) are constitutive relations represented in their inverse formulation and solved for the distortions.

Considering hence the Lagrange equations of (2.3) together with the definitions of momenta (3.4) and polymomenta (2.7), we obtain the last pair of quasi-canonical equations (5.6):

$$\begin{aligned} \dot{p}^\alpha_{(k)} - \bar{\nabla}_\beta \sigma^{\alpha\beta}_{(k)} &= -\frac{\partial \mathcal{H}^{DW}}{\partial u_\alpha^{(k)}}, \\ \dot{p}^3_{(k)} - \bar{\nabla}_\beta \sigma^{3\beta}_{(k)} &= -\frac{\partial \mathcal{H}^{DW}}{\partial u_3^{(k)}}. \end{aligned} \tag{5.6}$$

The equations (5.6) coincide with the equations (2.4) except the terms with $\sigma^{\alpha 3}_{(k)}$, $\sigma^{33}_{(k)}$ that are expressed through $u_\alpha^{(k)}$, $u_3^{(k)}$.

Finally, the natural boundary conditions (4.7) can be represented similarly to (2.5):

$$\begin{aligned} \left.\left(\sigma^{\alpha\beta}_{(k)} \nu_\beta - \frac{\partial \mathcal{H}^{DW}_\Gamma}{\partial u_\alpha^{(k)}}\right)\right|_{\partial S} &= 0; \\ \left.\left(\sigma^{3\beta}_{(k)} \nu_\beta - \frac{\partial \mathcal{H}^{DW}_\Gamma}{\partial u_\alpha^{(k)}}\right)\right|_{\partial S} &= 0 \end{aligned} \tag{5.7}$$

where formally $\mathcal{H}^{DW}_\Gamma = -\mathcal{L}_{\partial S}$ while the kinematic boundary conditions pair corresponds to the equations (5.5) in accordance with the approach shown in [39]. The initial conditions correspond to (4.8).

CONCLUSIONS

Starting from the Lagrangian formalism of the Vekua-Amosov general theory of thick plates, the conservation conditions are formulated, and the main motion integrals are constructed. It has to be noted that the presented plate model corresponds to the “elementary” theory that does not accounts for the boundary conditions on the faces of thin-walled structures. These boundary conditions are approximately satisfied as a result of the convergence of the two-dimensional solutions sequence to the solution of the three-dimensional initial-boundary value problem. Thus, the nonzero energy flux vector field and the normal component of the second-rank Hamilton tensor due to the boundary conditions discrepancy appear on the mid-plane; the total Hamiltonian H^0 as well as the integral field momentum vector components G_γ are therefore motion integrals under assumption of vanishing boundary conditions discrepancy as $N \to \infty$. In general, this drawback of “elementary” theories can be eliminated on the background of “extended” plate and shell theories (e. g. see [44-47]).

The Hamiltonian field equations for the Nth order plate theory are constructed. This formulation is variationally consistent and allows one the use of methods of lines in time domain for numerical solutions in transient plate dynamics’ problems. On the other hand, the instantaneous Hamiltonian formulation can be considered as an “evolutionary” system of ordinary differential equations in time domain, or, with coefficients containing differential operators. This formalism maybe useful in various semi-analytical approaches such as [31-34], [35] as well as [36-38].

The obtained de Donder – Weyl Hamiltonian $\mathcal{H}^0$ does not represent the total energy density [40] but allows one to construct the equations system that contains only first-order covariant derivatives. The obtained equations of Hamilton – de Donder – Weyl type (5.5) cannot be interpreted as canonical equations because of presence in their left hand sides the covariant derivatives; they can be rewritten nevertheless in the canonical representation by translating the terms with Christoffel symbols to their right hand sides. At the same time the dynamic equations (5.6) remain quasi-canonical due to the divergence operators and minus marks with the corresponding terms. The last drawback can be eliminated by means of different Hamiltonian construction (e. g. see [39]); the polymomenta definition also allows a certain degree of arbitrariness in the model formulation, so that the following general notation can be obtained:

$$\nabla^* \otimes \mathbf{U} = \frac{\partial \mathcal{H}}{\partial \mathbf{P}}, \quad \nabla^* \cdot \mathbf{P} = -\frac{\partial \mathcal{H}}{\partial \mathbf{U}},$$
$$\mathbf{P} = \{\mathbf{p} \quad \boldsymbol{\sigma}\}^{\mathrm{T}}, \quad \nabla^* = \{\partial_t \quad \bar{\nabla}\}.$$

The proposed Hamiltionian formulation can be considered only as a simplest one; its improvement become possible on the background of the powerful symplectic geometry formalism developed in the field theory [40, 41, 43]. Let us also note that the more complex Hamilton – Caratheodory formalism [41] offers nevertheless some features, at least in the numerical simulation, providing the use of Hamilton–Jacobi theory.

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