METHOD OF COMPENSATING LOADS FOR SOLVING OF A PROBLEM OF UNSYMMETRIC BENDING OF INFINITE ICE SLAB WITH CIRCULAR OPENING

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Abstract: Unsymmetric flexure of an infinite ice slab with circular opening is under examination. The mentioned construction is considered as an infinite plate of constant thickness resting on an elastic subgrade which properties are described by Winkler's model. The plate's thickness is variable in the area ajoining to the opening. Method of compensating loads is used. Basic and compensating solutions are received. The obtained solutions are produced in closed form in terms of Bessel functions.

Keywords: infinite ice slab, circular opening, method of compensating loads, Bessel functions

МЕТОД КОМПЕНСИРУЮЩИХ НАГРУЗОК ДЛЯ РЕШЕНИЯ ЗАДАЧИ О НЕСИММЕТРИЧНОМ ИЗГИБЕ БЕСКОНЕЧНОЙ ЛЕДЯНОЙ ПЛИТЫ С КРУГЛЫМ ОТВЕРСТИЕМ

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Аннотация: Рассматривается несимметричное изгиб бесконечной ледяной плиты с круглым отверстием. Указанное сооружение рассматривается как бесконечная пластинка постоянной толщины на упругом основании, свойства которого описываются моделью Винклера. Толщина плиты в области, примыкающей к отверстию, переменная. Для решения задачи используется метод компенсирующих нагрузок. С привлечением аппарата специальных функций (функции Бесселя) получены базовые и компенсирующие решения.

Ключевые слова: бесконечная ледяная плита, круглое отверстие, метод компенсирующих нагрузок, функции Бесселя

1. INTRODUCTION

The present work is dedicated to the strength analysis of an ice cover. Unsymmetric bending of an infinite ice slab of constant thickness is examined. The plate's thickness increases in the direction from an inner boundary in the area ajoining to the opening. As usual similar constructions are considered as plates resting on Winkler's elastic subgrade. Modulus of subgrade reaction is equal to the volume of water's weight. Related problems were considered in [1] and [2]. The present work gives another approach – the method of compensating loads. The proposed method will be more effective for plates with comparatively large openings analysis.

2. ANTISYMMETRIC BENDING OF INFINITE SLAB

First we will consider the outer part of the construction under investigation – the slab of constant thickness with circular opening (fig.1).

The above-mentioned structure is resting on an elastic subgrade and is subjected to an action of antisymmetric loading. Radius of the circular opening is x_1.

The plate is subjected to an action of antisymmetric loads

$$q(r,\theta) = q(r)\cos\theta \text{ or } q(r,\theta) = q(r)\sin\theta .$$

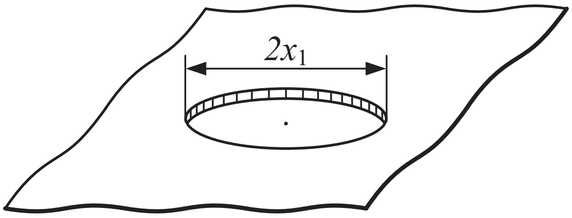

Figure 1. *Infinite plate with the circular opening.*

For example similar problems occur when wind loading influence is under study.

The differential equation of antisymmetric bending of a plate resting on Winkler's elastic subgrade is:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 w + \frac{c}{D}w = \frac{q}{D}, \tag{1}$$

where D is the flexural rigidity of a plate, c is the coefficient of soil reaction.

Let us introduce the following notation:

$$\ell = \sqrt[4]{\frac{D}{c}} \tag{2}$$

and going over dimensionless coordinate

$$x = \frac{r}{\ell}. \tag{3}$$

With the substitutions (2) and (3), the equation (1) assumes the form:

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{x^2}\right)^2 w + w = \frac{q}{c}. \tag{4}$$

The solution of homogeneous equation corresponding to (4) is sought in the shape:

$$w = F_1(x)\left(A_1 \sin\varphi + B_1\cos\varphi\right).$$

The function $F_1(x)$ satisfies the ordinary differential equation:

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{1}{x^2}\right)^2 F_1 + F_1 = 0. \tag{5}$$

The equation (5) can be reduced to the system of two ordinary conjugate linear equations:

$$\frac{d^2F_1}{dx^2} + \frac{1}{x}\frac{dF_1}{dx} - \frac{F_1}{x^2} \pm iF_1 = 0. \tag{6}$$

It has been known [3] that the solution of the equation (6) is expressed in terms of cylindrical functions

$$\begin{aligned} w = {} & \left(a_1\sin\varphi + b_1\cos\varphi\right)u_1(x) + \\ & + \left(c_1\sin\varphi + d_1\cos\varphi\right)\upsilon_1(x) + \\ & + \left(a_1'\sin\varphi + b_1'\cos\varphi\right)f_1(x) + \\ & + \left(c_1'\sin\varphi + d_1'\cos\varphi\right)g_1(x). \end{aligned} \tag{7}$$

With use of certain derivation formulae [3] we can obtain the following expressions for stresses:

$$\begin{aligned} M_1 = {} & -\frac{D}{\ell^2}\Bigg\{\left(a_1\sin\varphi + b_1\cos\varphi\right)\left[\upsilon_1(x) - \frac{1-\sigma}{x}\left(u_1'(x) - \frac{u_1(x)}{x}\right)\right] - \left(c_1\sin\varphi + d_1\cos\varphi\right)\times \\ & \times\left[u_1(x) + \frac{1-\sigma}{x}\left(\upsilon_1'(x) - \frac{\upsilon_1(x)}{x}\right)\right] + \left(a_1'\sin\varphi + b_1'\cos\varphi\right)\left[g_1(x) - \frac{1-\sigma}{x}\left(f_1'(x) - \frac{f_1(x)}{x}\right)\right] - \\ & - \left(c_1'\sin\varphi + d_1'\cos\varphi\right)\left[f_1(x) + \frac{1-\sigma}{x}\left(g_1'(x) - \frac{g_1(x)}{x}\right)\right]\Bigg\}, \end{aligned} \tag{8}$$

$$M_2 = -\sigma \frac{D}{\ell^2}\left\{(a_1 \sin\varphi + b_1 \cos\varphi)\left[\upsilon_1(x) + \frac{1-\sigma}{\sigma x}\left(u_1'(x) - \frac{u_1(x)}{x}\right)\right] + (c_1 \sin\varphi + d_1 \cos\varphi)\times\right.$$
$$\times\left[-u_1(x) + \frac{1-\sigma}{\sigma x}\left(\upsilon_1'(x) - \frac{\upsilon_1(x)}{x}\right)\right] + (a_1'\sin\varphi + b_1'\cos\varphi)\left[g_1(x) + \frac{1-\sigma}{\sigma x}\left(f_1'(x) - \frac{f_1(x)}{x}\right)\right] + \tag{9}$$
$$\left. + (c_1'\sin\varphi + d_1'\cos\varphi)\left[-f_1(x) + \frac{1-\sigma}{\sigma x}\left(g_1'(x) - \frac{g_1(x)}{x}\right)\right]\right\},$$

$$H_1 = -H_2 = (1-\sigma)\frac{D}{\ell^2 x}\left\{(a_1 \cos\varphi - b_1 \sin\varphi)\left[u_1'(x) - \frac{u_1(x)}{x}\right] + \right.$$
$$+ (c_1 \cos\varphi - d_1 \sin\varphi)\left[\upsilon_1'(x) - \frac{\upsilon_1(x)}{x}\right] + (a_1'\cos\varphi - b_1'\sin\varphi)\left[f_1'(x) - \frac{f_1(x)}{x}\right] + \tag{10}$$
$$\left. + (c_1'\cos\varphi - d_1'\sin\varphi)\left[g_1'(x) - \frac{g_1(x)}{x}\right]\right\},$$

$$Q_1 = -\frac{D}{\ell^3}[(a_1 \sin\varphi + b_1 \cos\varphi)\upsilon_1'(x) - (c_1 \sin\varphi + d_1 \cos\varphi)u_1'(x) +$$
$$+ (a_1'\sin\varphi + b_1'\cos\varphi)g_1'(x) - (c_1'\sin\varphi + d_1'\cos\varphi)f_1'(x)], \tag{11}$$

$$Q_2 = -\frac{D}{x\ell^3}[(a_1 \cos\varphi - b_1 \sin\varphi)\upsilon_1(x) - (c_1 \cos\varphi - d_1 \sin\varphi)u_1(x) +$$
$$+ (a_1'\cos\varphi - b_1'\sin\varphi)g_1(x) - (c_1'\cos\varphi - d_1'\sin\varphi)f_1(x)]. \tag{12}$$

3. METHOD OF COMPENSATING LOADS

To resolve the problem under consideration the method of compensating loads is applied. The result to be sought is represented as a sum of a basic solution w_0 and a compensating one w_k:

$$w = w_0 + w_k. \tag{13}$$

The basic solution satisfies to the differential equation describing the problem under study. The solution w_0 has singularities which are relevant to the acting loads. For example for the case of a plate subjected to an action of a concentrated force the sought basic solution w_0 must have peculiarities of a concentrated force. However the basic solution does not satisfy boundary conditions.

The sought compensating solution w_k complies the homogeneous differential equation for the plate's domain. It is required that the sum of w_0 and w_k must satisfy boundary conditions.

4. THE DEVELOPMENT OF BASIC SOLUTION

The decision of the problem of infinite plate resting on an elastic subgrade and subjected to an action of concentrated force is taken as basic solution:

$$w_0 = \frac{P\ell^2}{4D} f_0(x). \tag{14}$$

The expression (14) is the main influence function when $P = 1$. We can receive basic solutions for certain particular problems by integrating of the formula (14).

The plate subjected to an action of the forces $q\cos\theta$ distributed along circumference with the reduced radius is under consideration. The principle of composition of actions is utilized.

The plate's deflection in the point with the coordinates x, y can be represented in the following form:

$$w = \frac{q\alpha\ell^3}{4D} \times \\ \times \int_0^{2\pi} f_0\left(\sqrt{\alpha^2 + x^2 - 2\alpha x\cos(\theta - \varphi)}\right)\cos\theta d\theta. \tag{15}$$

For calculation of (15) we will use the formula of cylindrical functions compositions

$$Z_0\left(\sqrt{\alpha^2 + x^2 - 2\alpha x\cos(\theta - \varphi)}\right) = \\ = 2\sum_{n=0}^{n}{}' J_n(\alpha)Z_n(x)\cos n(\theta - \varphi), \tag{16}$$

where the sign ' denotes that when $n = 0$ we introduce the coefficient $\frac{1}{2}$. The expression (16) is fulfilled when $\alpha < x$. When $\alpha > x$ we must interchange the position of α and x in the right-hand part of (16).
Let us assume that

$$Z_0 = H_0^{(1)}\left(\sqrt{i}\sqrt{\alpha^2 + x^2 - 2\alpha x\cos(\theta - \varphi)}\right),$$

we integrate and separate the real part. Then we obtain:
when $x \le \alpha$

$$w = w_I = \frac{\pi\alpha q\ell^3}{2D}[u_1(x)f_1(\alpha) - \\ - \upsilon_1(x)g_1(\alpha)]\cos\varphi, \tag{17}$$

when $x \ge \alpha$

$$w = w_{II} = \frac{\pi\alpha q\ell^3}{2D}[u_1(\alpha)f_1(x) - \\ - \upsilon_1(\alpha)g_1(x)]\cos\varphi. \tag{18}$$

Let us consider a plate subjected to an action of the loading

$$q = a_1\cos\theta + b_1\sin\theta,$$

which is distributed along the circumference with the reduced radius α.
By use the formulae (17), (18) we receive the solution:
when $\alpha \le x$

$$w = \frac{\pi\alpha\ell^3}{2D}\{a_1[u_1(\alpha)f_1(x) - \upsilon_1(\alpha)g_1(x)] \times \\ \times \sin\varphi + b_1[u_1(\alpha)f_1(x) - \upsilon_1(\alpha)g_1(x)] \times \\ \times \cos\varphi\}, \tag{19}$$

when $\alpha \ge x$

$$w = \frac{\pi\alpha\ell^3}{2D}\{a_1[u_1(x)f_1(\alpha) - \upsilon_1(x)g_1(\alpha)] \times \\ \times \sin\varphi + b_1[u_1(x)f_1(\alpha) - \upsilon_1(x)g_1(\alpha)] \times \\ \times \cos\varphi\}. \tag{20}$$

Using the theory of Bessel functions we can receive basic solutions for different kinds of loads. For instance we can obtain the solution for the loads distributed over ring surface by integrating the expressions (17) and (18).

5. THE COMPENSATING SOLUTION

The compensating solution will be represented in the following form:

$$w_k = (a_1'\sin\varphi + b_1'\cos\varphi)f_1(x) + \\ + (c_1'\sin\varphi + d_1'\cos\varphi)g_1(x). \tag{21}$$

We assume that the origin of coordinates is in the midpoint of an opening. The radius of the opening is equal to x_1. Therefore the solution can contain only finite values of Bessel functions f, g and their derivatives. The domain under consideration does not include the midpoint $x = 0$ where the Bessel functions f and g have certain singularities.
The compensating solution for the plate with free circular opening is:

$$w_k = -\frac{\ell^2}{D}\frac{1}{f_1^{(Q)}(x_1)g_1^{(M)}(x_1) - f_1^{(M)}(x_1)g_1^{(Q)}(x_1)}\{f_1(x)[(\ell N_1 f_1^{(M)}(x_1) - K_1 f_1^{(Q)}(x_1))\sin\varphi + \\ + (\ell O_1 f_1^{(M)}(x_1) - L_1 f_1^{(Q)}(x_1))\cos\varphi] - g_1(x)[(\ell N_1 g_1^{(M)}(x_1) - K_1 g_1^{(Q)}(x_1))\sin\varphi + \\ + (\ell O_1 g_1^{(M)}(x_1) - L_1 g_1^{(Q)}(x_1))\cos\varphi]\}. \quad (22)$$

The notation for $f_1^{(M)}(x), g_1^{(M)}(x), f_1^{(Q)}(x), g_1^{(Q)}(x)$ are given in [3].

6. CALCULATION OF THE INTERIOR PART OF THE PLATE HAVING VARIABLE THICKNESS

An inner part of the plate adjoining to the circular opening is examined. This part has variable thickness increasing along the direction from the internal boundary. This part is considered as a ring plate with inner radius x_A and the outer one x_1.

The differential equation describing the antisymmetric flexure of the circular plate of variable thickness resting on elastic Winkler's subgrade is:

$$D\nabla^2\nabla^2 w + \frac{dD}{dx}\left[2\frac{d^3w}{dx^3} + \frac{2+\sigma}{x}\frac{d^2w}{dx^2} - \\ - \frac{3}{x^2} + \frac{3}{x^3}w\right] + \frac{d^2D}{dx^2}\left[\frac{d^2w}{dx^2} + \\ + \sigma\left(\frac{1}{x}\frac{dw}{dx} - \frac{1}{x^2}w\right)\right] = (q_{акт} - cw)r_0^4, \quad (23)$$

where $x = \frac{r}{r_0}$, r_0 - constant.

Let us assume that the flexural rigidity varies according to the power law:

$$D = D_0 x^m, \quad (24)$$

where D_0 is the constant.

The producted analysis showed that for the problem under study there is no way to obtain solutions in closed form in particular in terms of Bessel functions. This fact is distinct from the case of the similar plate symmetric bending. The exception is the case when in (24) $m = 4$. Then the equation (23) by means of certain substitutions is reduced to the equation with constant coefficients of Euler class. The mentioned equation has the solution expressed in terms of power functions:

$$w = x^{-1}\left(A_1x^{\alpha_1} + A_2x^{\alpha_2} + A_3x^{\alpha_3} + A_4x^{\alpha_4}\right)\times \\ \times\cos\theta, \quad (25)$$

where

$$\alpha_{1,2,3,4} = \pm\sqrt{2(2-\sigma) \pm \sqrt{2\sigma^2 - \beta^4}}, \quad (26)$$

$$\beta^4 = \frac{cr_0^4}{D_0}.$$

Furthermore the conditions of conjugation of two parts when $x = x_1$ are to be fulfilled. For this aim the deflections, angles, bending moments, transversal forces of inner and outer parts are equating.

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