DOI:10.22337/2587-9618-2019-15-3-131-148

ANALYSIS OF FORCED VIBRATIONS OF NONLINEAR PLATES IN A VISCOELASTIC MEDIUM UNDER THE CONDITIONS OF THE DIFFERENT COMBINATIONAL INTERNAL RESONANCES

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Abstract: In the present paper, the force driven dynamic response of a nonlinear plate embedded in a viscoelastic medium, damping features of which are described by the Kelvin-Voigt fractional derivative model, is studied. The motion of the plate is described by three coupled nonlinear differential equations with due account for the fact that the plate is being under the conditions of the internal combinational resonance accompanied by the external resonance, resulting in the interaction of three modes corresponding to the mutually orthogonal displacements. A comparative analysis of numerical calculations for the cases of free and forced vibrations has been carried out.

Keywords: Nonlinear vibrations of thin plates, interaction of internal and external resonances, fractional derivative viscoelastic surrounding medium, combinational internal resonance

ЧИСЛЕННЫЙ АНАЛИЗ ВЫНУЖДЕННЫХ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ ПЛАСТИНОК В ВЯЗКОУПРУГОЙ СРЕДЕ ПРИ НАЛИЧИИ КОМБИНАЦИОННОГО РЕЗОНАНСА

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Аннотация: Исследованы нелинейные вынужденные колебания тонких пластинок в вязкоупругой среде, демпфирующие свойства которой задаются с помощью модели Кельвина-Фойгта с дробной производной. Колебания пластинки в вязкоупругой среде описываются в декартовской системе координат тремя дифференциальными уравнениями, с учетом того, что пластинка находится в условиях внутреннего комбинационного резонанса, сопровождаемого внешним резонансом. Приведен сравнительный анализ численных исследований свободных и вынужденных колебаний при наличии различных комбинационных внутренних резонансов для различных геометрических параметров пластинки.

Ключевые слова: нелинейные колебания пластинок, сочетание внутреннего и внешнего резонансов, дробная производная, комбинационный резонанс

1. INTRODUCTION

Recently the interest to nonlinear dynamic response of viscoelastic plates or elastic plates vibrating in a viscoelastic surrounding medium has been greatly renewed due to the appearance of advanced materials exhibiting nonlinear behaviour, and a comprehensive review in the field, including experimental results, could be found in [1-6]. In so doing the damping forces are usually taken into account via the Rayleigh's hypothesis [1,7], resulting in the modal damping [8], i.e. it is assumed that each natural mode of vibrations possesses its own damping coefficient dependent on its natural frequency. For describing the viscoelastic features of plates, the Kelvin-Voigt model [4] or standard linear solid model [5] are of frequent use in engineering practice considering either linear or nonlinear springs in viscoelastic elements [9].

The analysis of free undamped [10] and damped [4] vibrations of nonlinear systems is of great importance for defining the dynamic system's characteristics dependent on the amplitude-phase relationships and modes of vibration. Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in strong coupling between the modes of vibrations involved [10-15] and hence in the energy exchange between the interacting modes.

The internal resonance could be observed in the case of some combination of natural frequencies of one and the same type of vibrations. Thus, nonlinear vibrations of rectangular plates, dynamic behaviour of which is described by von Karman equations in terms of the plate's deflection and stress function, have been considered in [12] by reducing the governing equations to a set of two modal equations applying the Galerkin procedure. The case of the one-to-one internal resonance (when frequencies of two modes of flexural vibration are equal to each other) accompanied by the external resonance (when the frequency of the harmonic force is close to one of the natural frequency) has been studied.

The one-to-one internal resonance has been investigated also in [13] and [14] for nonlinear vertical vibrations of rectangular plates under the action of harmonic forces acting in the plate's plane [13] and out of the plate's plane [13,14], in so doing a set of three equations in terms of two in-plane displacements and deflection and a set of five equations considering the shear deformations have been used in [13] and [14], respectively. However, considering the inertia forces only for vertical vibrations and utilizing the Galerkin procedure, in both papers a set of two nonlinear equations has been obtained in terms of two flexural modes, which are assumed to be coupled via the one-to-one internal resonance.

For the first two natural modes of flexural vibrations, the cases of the 1:2 and 1:3 internal resonances have been also studied in [14].

Another type of the internal resonance has been investigated by Rossikhin and Shitikova [15-18], when one frequency of in-plane vibrations is equal (the 1:1 internal resonance [17,18]) or two times larger (the 1:2 internal resonance [15,18]) than a certain frequency of out-of-plane vibrations. As this takes place, a set of three nonlinear differential equations in terms of three mutually orthogonal displacements has been used considering inertia of all types of vibrations, what allows the authors to study further the combinational resonances of the additive and difference types [16,19-20]. Combinational types of the internal resonance result in the energy exchange between three or more subsystems. It should be noted that investigations in this direction were initiated by Witt and Gorelik [21], who pioneered in the theoretical and experimental analysis of the energy transfer from one subsystem to another using the simplest two-degree-of-freedom mechanical system, as an example.

Moreover, in order to study nonlinear free damped vibrations of a thin plate, the viscoelastic Kelvin-Voigt model involving fractional derivative [22] has been utilized, since this model possesses the advantage over the conventional Kelvin-Voigt model [10-14], because it provides the results matching the experimental data. Thus, for example, experimental data on ambient vibrations study for the Vincent-Thomas [23] and Golden Gate [24] suspension bridges have shown that different modes of vibrations possess different magnitudes of damping coefficients. Besides, the increase in the natural frequency results in the decrease in the damping ratio. In order to lead the theoretical investigation in the agreement with the experiment, in 1998 it was suggested in [25] to utilize the fractional derivatives to describe the processes of internal friction occurring in suspension combined systems, what allowed the authors in a natural way to obtain the damping ratios, which depend on natural frequencies.

Nowadays fractional calculus is widely used for solving linear and nonlinear dynamic problems of structural mechanics, what is evident from numerous studies in the field, the overview of which could be found in the state-of-the-art articles by Rossikhin and Shitikova [26,27], wherein the examples of adopting the fractional derivative Kelvin-Voigt, Maxwell and standard linear solid models are provided for single-mass oscillators, rods, beams, plates, and shells.

In particular, linear vibrations of Kirchhoff-Love plates with the Kelvin-Voigt fractional damping were considered for rectangular and circular plates, respectively, in [28] and [29] using one equation for vertical vibrations, while utilizing three equations of in-plane and transverse vibrations in [7,30], and later multiplate systems were analyzed in [26,31]. It has been proved [27,32] that if viscoelastic properties of plates are described by the Kelvin-Voigt model assuming the Poisson's ratio as the time-independent value (though for real viscoelastic materials the Poisson's ratio is always a time-dependent function [33]), then this case coincides with the case of the dynamic behaviour of elastic bodies in a viscoelastic medium. Thus, the authors of [28,29], and not only them, replaced one problem with another, namely: a problem of the dynamic response of viscoelastic Kirchhoff-Love plates in a conventional medium with a problem of dynamic response of elastic Kirchhoff-Love plates in a viscoelastic medium, damping features of which are governed by the fractional derivative Kelvin-Voigt model. The vibration suppression of fractionally damped thin rectangular simply supported plates subjected to a concentrated harmonic loading has been studied recently in [34] in order to minimize the plate deflection, in so doing the vibration suppression is accomplished by attaching multiple absorbers modelled as the Kelvin-Voigt fractional oscillators, i.e. generalizing the approach suggested in [26,31].

As for the analysis of nonlinear vibrations of plates, then except the above mentioned papers [15-20], the fractional derivative Kelvin-Voigt model was used in [35-40] and fractional derivative standard linear solid model in [6,41,42] but without considering the phenomena of the internal resonance. Thus, free and forced vertical vibrations of an orthotropic plate have been studied in [35] considering first four modes of flexural vibrations, and during the analysis of force driven vibrations the frequency of a harmonic force was assumed to be equal to one of natural frequencies. The von Karman plate equation with fractional derivative damping was utilized in [36] for analyzing the cases of primary, subharmonic and superharmonic resonance conditions, when the harmonic force frequency, respectively, is approximately equal, three times less or larger than the first or second natural frequency of vertical vibrations. Nonlinear random vibrations of the same plate was studied in [39]. Dynamic nonlinear response to random excitation of a simply supported rectangular plate based on a foundation, damping features of which are described by the fractional derivative Kelvin-Voigt model, has been considered in [38]. The analysis of chaotic vibrations of simply supported nonlinear viscoelastic plate with fractional derivative Kelvin-Voigt model has been carried out in [40] for the case when the plate is subjected to an in-plane harmonic force in one direction and a transverse harmonic force. The Galerkin decomposition has been used to obtain the modal equation of the system, in so doing the authors restricted themselves only by the first mode. The fractional derivative standard linear solid model has been utilized in [42] for a viscoelastic layer for active damping of geometrically nonlinear vibrations of smart composite plates using the higher order plate theory and finite element method with discretizing the plate by eight-node isoparametric quadrilateral elements.

In the present paper, the approaches suggested in [18] for solving the problem on free nonlinear vibrations of elastic plates in a viscoelastic medium, damping features of which are governed by the Riemann-Liouville derivatives of the fractional order, and in [43] for studying the dynamic response of the fractional Duffing oscillator subjected to harmonic loading are generalized for the case of forced vibrations of a simply-supported nonlinear thin elastic plate under the conditions of different combinational internal resonances, when three natural modes corresponding to mutually orthogonal displacements are coupled.

2. PROBLEM FORMULATION

2.1. Governing equations.

Let us consider the dynamic behavior of a simply supported nonlinear thin rectangular plate, vibrations of which in a viscoelastic fractional derivative medium are described by the following three differential equations in the dimensionless form [44, 45]:

$$u_{xx} + \frac{1-\nu}{2}\beta_1^2 u_{yy} + \frac{1+\nu}{2}\beta_1 v_{xy} + \frac{1+\nu}{2}\beta_1^2 w_y w_{xy} + \\ + w_x\left(w_{xx} + \frac{1-\nu}{2}\beta_1^2 w_{yy}\right) = \ddot{u} + æ_1 D_{0+}^{\gamma} u, \tag{1}$$

$$\beta_1^2 v_{yy} + \frac{1-\nu}{2} v_{xx} + \frac{1+\nu}{2}\beta_1 u_{xy} + \frac{1+\nu}{2}\beta_1 w_x w_{xy} + \\ + \beta_1 w_y\left(\beta_1^2 w_y + \frac{1-\nu}{2} w_{xx}\right) = \ddot{v} + æ_2 D_{0+}^{\gamma} v, \tag{2}$$

$$\frac{\beta_2^2}{12}(w_{xxxx} + 2\beta_1^2 w_{xxyy} + \beta_1^4 w_{yyyy}) - \\ - w_{xx}(u_x + \nu\beta_1 v_y) - w_x(u_{xx} + \nu\beta_1 v_{xy}) - \\ - \frac{1-\nu}{2}\beta_1\left[w_{xy}(\beta_1 u_y + v_x) + w_y(\beta_1 u_{xy} + v_{xx})\right] - \\ - \beta_1^2\left[w_{yy}(\nu u_x + \beta_1 v_y) + w_y(\nu u_{xy} + \beta_1 v_{yy})\right] - \\ - \frac{1-\nu}{2}\beta_1\left[w_{xy}(\beta_1 u_y + v_x) + w_x(\beta_1 u_{yy} + v_{xy})\right] - \\ - \hat{F}\delta(x-x_0)\delta(y-y_0)\cos(\Omega_F t) = \tag{3}$$

$$= -\ddot{w} - æ_3 D_{0+}^{\gamma} w, \tag{4}$$

where $u = u(x,y,t)$, $v = v(x,y,t)$, and $w = w(x,y,t)$ are the displacements of points located in the plate's middle surface in the *x*-, *y*-, and *z*-directions, respectively, ν is the Poisson's ratio, $\beta_1 = a/b$ and $\beta_2 = h/a$ are the parameters defining the dimensions of the plate, *a* and *b* are the plate's dimensions along the *x*- and *y*-axes, respectively, *h* is the thickness, *t* is the time,

$$F = \hat{F}\delta(x-x_0)\delta(y-y_0)\cos(\Omega_F t)$$

is the harmonic force applied at the point with the coordinates x_0, y_0, $\hat{F}$ and Ω_F are its amplitude and frequency, respectively, δ is the Dirac delta function,

$$æ_i = \varepsilon\mu_i\tau_i^{\gamma}\ (i = 1,2,3)$$

are damping coefficients, ε is a small dimensionless parameter of the same order of magnitude as the amplitudes, μ_i are finite values, τ_i is the relaxation time of the *i*th generalized displacement, D_{0+}^{γ} is the Riemann-Liouville fractional derivative of the γ-order [46], an overdot denotes the time-derivative, and lower indices label the derivatives with respect to the corresponding coordinates.

For solving nonlinear governing equations of motion (1)-(3), the procedure resulting in decoupling linear parts of equations has been proposed with the further utilization of the method of multiple scales [18,44,45], in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales. It has been shown that the phenomenon of internal resonance could be very critical, since in the thin plate under consideration the internal resonance is always present. Moreover, its type depends on the order of smallness of the viscosity involved into consideration [18]. The

following types of the internal resonance have been revealed:
of the order of ε:
the two-to-one internal resonance (1:2)

$$\begin{aligned} &\omega_1 = 2\omega_3 \quad (\omega_2 \neq \omega_1, \omega_2 \neq 2\omega_3), \\ &\omega_2 = 2\omega_3 \quad (\omega_1 \neq \omega_2, \omega_1 \neq 2\omega_3); \end{aligned} \tag{5}$$

the one-to-one-to-two internal resonance $(1:1:2)$

$$\omega_1 = \omega_2 = 2\omega_3; \tag{6}$$

of the order of ε^2:
the one-to-one internal resonance (1:1)

$$\begin{aligned} &\omega_1 = \omega_2 \quad (\omega_3 \neq \omega_1, \omega_3 \neq \omega_2), \\ &\omega_1 = \omega_3 \quad (\omega_2 \neq \omega_1, \omega_2 \neq \omega_3), \\ &\omega_2 = \omega_3 \quad (\omega_1 \neq \omega_2, \omega_1 \neq \omega_3); \end{aligned} \tag{7}$$

the one-to-one-to-one internal resonance $(1:1:1)$

$$\omega_1 = \omega_2 = \omega_3; \tag{8}$$

the combinational resonance of the additive-difference type

$$2\omega_3 = \omega_1 + \omega_2, \tag{9}$$

$$2\omega_3 = \omega_1 - \omega_2, \tag{10}$$

$$2\omega_3 = \omega_2 - \omega_1, \tag{11}$$

where ω_1 and ω_2 are the frequencies of certain modes of in-plane vibrations in the *x*- and *y*-axes, respectively, and ω_3 is the frequency of a certain mode of vertical vibrations.
Note that the cases of the internal resonances (4)-(7) have been studied recently by the authors in [44,45,47,48]. Thus, below we will examine in detail all possible cases of the combinational resonances (8)-(10).

2.2. Combinational resonance of the additive type $2\omega_3 \simeq \omega_1 + \omega_2$

Now let us consider the case of the additive internal combinational resonance (8) accompanied by the external resonance, i.e.,

$$2\omega_3 = \omega_1 + \omega_2 + 2\varepsilon^2\sigma$$

and

$$\Omega_F = \omega_3 + \varepsilon^2\sigma_F,$$

where σ is the detuning parameter characterizing the nearness between the natural frequencies of the coupled modes, and σ_F is the second detuning parameter defining the difference between the frequency of vertical vibrations and the frequency of the external force Ω_F.
Using the set of solvability equations to eliminate secular terms similarly to the case of free vibrations considered in [18] and adding the external resonance, we obtain the following solvability equations for the case of force driven vibrations:

$$\begin{aligned} &2i\omega_1 D_2 A_1 + \mu_1 (i\omega_1\tau_1)^\gamma A_1 + 2\zeta_1 (k_5 + k_7) A_1 A_3 \bar{A}_3 + \\ &+2\zeta_1 k_8 \bar{A}_2 A_3^2 \exp(2i\sigma T_2) = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &2i\omega_2 D_2 A_2 + \mu_2 (i\omega_2\tau_2)^\gamma A_2 + 2\zeta_2 (k_6 + k_8) A_2 A_3 \bar{A}_3 + \\ &+2\zeta_2 k_7 \bar{A}_1 A_3^2 \exp(2i\sigma T_2) = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &2i\omega_3 D_2 A_3 + \mu_3 (i\omega_3\tau_3)^\gamma A_3 + \\ &+\left[\zeta_{13}(k_1 + 2k_2) + \zeta_{23}(k_3 + 2k_4)\right] A_3^2 \bar{A}_3 + \\ &+\zeta_{13}(k_5 + k_7) A_1 \bar{A}_1 A_3 + \zeta_{23}(k_6 + k_8) A_2 \bar{A}_2 A_3 + \\ &+(\zeta_{13} k_8 + \zeta_{23} k_7) A_1 A_2 \bar{A}_3 \exp(-2i\sigma T_2) - \\ &-2f \exp(i\sigma_F T_2) = 0, \end{aligned} \tag{14}$$

where $D_2 = \partial / \partial T_2$ is the time-derivative due to the utilization of the generalized method of multiple time scales [18], $A_j(T_2)$ $(j=1,2,3)$ are unknown complex functions, $\zeta_1, \zeta_2, \zeta_{13}, \zeta_{23}$ are coefficients depending on the plate dimensions and numbers of excited modes [18], $k_p\ (p=1,2,...8)$ are coefficients depending on the natural frequencies of plate, and f is a finite value.

To eliminate $\exp(\pm 2i\sigma_1 T_2)$ from equations (11)-(13), let us introduce the substitution

$$A_3(T_2) = A_3\exp(-i\sigma T_2). \tag{15}$$

Representing the functions $A_i(T_2)$ in equations (11)-(13) in the polar form

$$A_i(T_2) = a_i(T_2)\exp\left[i\varphi_i(T_2)\right] \quad (i=1,2,3)$$

and separating real and imaginary parts yield

$$\left(a_1^2\right)^{\cdot} + s_1 a_1^2 = -2\omega_1^{-1}\zeta_1 k_8 a_1 a_2 a_3^2 \sin\delta, \tag{16}$$

$$\left(a_2^2\right)^{\cdot} + s_2 a_2^2 = -2\omega_2^{-1}\zeta_2 k_7 a_1 a_2 a_3^2 \sin\delta, \tag{17}$$

$$\begin{aligned}\left(a_3^2\right)^{\cdot} + s_3 a_3^2 &= -2f\omega_3^{-1}a_3\sin\beta_a + \\ &+\omega_3^{-1}\left(\zeta_{13}k_8 + \zeta_{23}k_7\right)a_1 a_2 a_3^2 \sin\delta,\end{aligned} \tag{18}$$

$$\begin{aligned}\dot\varphi_1 &= \frac{1}{2}\lambda_1 + \omega_1^{-1}\zeta_1\left(k_5+k_7\right)a_3^2 + \\ &+\omega_1^{-1}\zeta_1 k_8 a_1^{-1} a_2 a_3^2 \cos\delta,\end{aligned} \tag{19}$$

$$\begin{aligned}\dot\varphi_2 &= \frac{1}{2}\lambda_2 + \omega_2^{-1}\zeta_2\left(k_6+k_8\right)a_3^2 + \\ &+\omega_2^{-1}\zeta_2 k_7 a_1 a_2^{-1} a_3^2 \cos\delta,\end{aligned} \tag{20}$$

$$\begin{aligned}\dot\varphi_3 &= \frac{1}{2}\lambda_3 + \frac{1}{2}\omega_3^{-1}\zeta_{13}\left(k_5+k_7\right)a_1^2 + \\ &+\frac{1}{2}\omega_3^{-1}\zeta_{23}\left(k_6+k_8\right)a_2^2 + \\ &+\frac{1}{2}\omega_3^{-1}\left[\zeta_{13}\left(k_1+2k_2\right)+\zeta_{23}\left(k_3+2k_4\right)\right]a_3^2 + \\ &+\frac{1}{2}\omega_3^{-1}\left(\zeta_{13}k_8+\zeta_{23}k_7\right)a_1 a_2\cos\delta - \\ &-f\left(\omega_3 a_3\right)^{-1}\cos\beta_a + \sigma,\end{aligned} \tag{21}$$

where a dot denotes differentiation with respect to T_2, a_i and φ_i are amplitudes and phases, respectively,

$$\delta = 2\varphi_3 - \varphi_2 - \varphi_1$$

is the phase difference,

$$s_i = \mu_i\tau_i^\gamma\omega_i^{\gamma-1}\sin\psi, \quad \psi = \pi\gamma/2,$$
$$\lambda_i = \mu_i\tau_i^\gamma\omega_i^{\gamma-1}\cos\psi, \text{ and } \beta_a = \varphi_3 - (\sigma_F+\sigma)T_2.$$

The set of Eqs. (15)-(20) describes the phase-amplitude modulations at nonlinear forced vibrations (1)-(3) in the case of the additive combinational resonance (8), and it is the generalizations of the case of free vibrations considered in detail in [19].

2.3. Combinational resonances of the difference type $2\omega_3 \simeq \omega_1 - \omega_2$

Now let us consider the difference combinational resonance (9) accompanied by the external resonance, i.e. when

$$2\omega_3 = \omega_1 - \omega_2 + 2\varepsilon^2\sigma$$

and

$$\Omega_F = \omega_3 + \varepsilon^2\sigma_F.$$

Then eliminating secular terms, we obtain the following solvability equations:

$$\begin{aligned}&2i\omega_1 D_2 A_1 + \mu_1\left(i\omega_1\tau_1\right)^\gamma A_1 + 2\zeta_1\left(k_5+k_7\right)A_1 A_3\bar A_3 + \\ &+2\zeta_1 k_6 A_2 A_3^2\exp\left(2i\sigma T_2\right) = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}&2i\omega_2 D_2 A_2 + \mu_2\left(i\omega_2\tau_2\right)^\gamma A_2 + 2\zeta_2\left(k_6+k_8\right)A_2 A_3\bar A_3 + \\ &+2\zeta_2 k_7 A_1\bar A_3^2\exp\left(-2i\sigma T_2\right) = 0,\end{aligned} \tag{23}$$

$$2i\omega_3 D_2 A_3 + \mu_3 \left(i\omega_3\tau_3\right)^{\gamma} A_3 +$$
$$\left[\zeta_{13}\left(k_1+2k_2\right)+\zeta_{23}\left(k_3+2k_4\right)\right]A_3^2\bar{A}_3 +$$
$$+\zeta_{13}\left(k_5+k_7\right)A_1\bar{A}_1A_3+\zeta_{23}\left(k_6+k_8\right)A_2\bar{A}_2A_3 + \quad (24)$$
$$+\left(\zeta_{13}k_6+\zeta_{23}k_7\right)A_1\bar{A}_2\bar{A}_3\exp\left(-2i\sigma T_2\right)-$$
$$-2f\exp\left(i\sigma_F T_2\right)=0.$$

Applying to (21)-(23) the same procedure as it has been done above for (11)–(13), as a result, we have

$$\left(a_1^2\right)^{\cdot} + s_1a_1^2 = -2\omega_1^{-1}\zeta_1k_6a_1a_2a_3^2\sin\delta, \quad (25)$$

$$\left(a_2^2\right)^{\cdot} + s_2a_2^2 = 2\omega_2^{-1}\zeta_2k_7a_1a_2a_3^2\sin\delta, \quad (26)$$

$$\left(a_3^2\right)^{\cdot} + s_3a_3^2 = -2f\omega_3^{-1}a_3\sin\beta_a + \quad (27)$$
$$+\omega_3^{-1}\left(\zeta_{13}k_6+\zeta_{23}k_7\right)a_1a_2a_3^2\sin\delta,$$

$$\dot{\varphi}_1 = \frac{1}{2}\lambda_1 + \omega_1^{-1}\zeta_1\left(k_5+k_7\right)a_3^2 + \quad (28)$$
$$+\omega_1^{-1}\zeta_1k_6a_1^{-1}a_2a_3^2\cos\delta,$$

$$\dot{\varphi}_2 = \frac{1}{2}\lambda_2 + \omega_2^{-1}\zeta_2\left(k_6+k_8\right)a_3^2 + \quad (29)$$
$$+\omega_2^{-1}\zeta_2k_7a_1a_2^{-1}a_3^2\cos\delta,$$

$$\dot{\varphi}_3 = \frac{1}{2}\lambda_3 + \frac{1}{2}\omega_3^{-1}\zeta_{13}\left(k_5+k_7\right)a_1^2 +$$
$$+\frac{1}{2}\omega_3^{-1}\zeta_{23}\left(k_6+k_8\right)a_2^2 + \sigma_1 +$$
$$+\frac{1}{2}\omega_3^{-1}\left[\zeta_{13}\left(k_1+2k_2\right)+\zeta_{23}\left(k_3+2k_4\right)\right]a_3^2 +$$
$$+\frac{1}{2}\omega_3^{-1}\left(\zeta_{13}k_6+\zeta_{23}k_7\right)a_1a_2\cos\delta - \quad (30)$$
$$-f\left(\omega_3a_3\right)^{-1}\cos\beta_a,$$

where $\delta = 2\varphi_3+\varphi_2-\varphi_1$

is the phase difference.

The set of Eqs. (24)-(29) describes the phase-amplitude modulations at nonlinear forced vibrations (1)-(3) in the case of the difference combinational resonance (9).

2.4. Combinational resonances of the difference type $2\omega_3 \simeq \omega_2-\omega_1$

Now let us consider the difference combinational resonance (10) subjected to the external resonance, i.e. when

$$2\omega_3 = \omega_2-\omega_1+2\varepsilon^2\sigma$$

and

$$\Omega_F = \omega_3+\varepsilon^2\sigma_F .$$

In this case the solvability equations have the form

$$2i\omega_1D_2A_1+\mu_1\left(i\omega_1\tau_1\right)^{\gamma}A_1+2\zeta_1\left(k_5+k_7\right)A_1A_3\bar{A}_3+$$
$$+2\zeta_1k_8A_2\bar{A}_3^2\exp\left(-2i\sigma T_2\right)=0, \quad (31)$$

$$2i\omega_2D_2A_2+\mu_2\left(i\omega_2\tau_2\right)^{\gamma}A_2+2\zeta_2\left(k_6+k_8\right)A_2A_3\bar{A}_3+$$
$$+2\zeta_2k_5A_1A_3^2\exp\left(2i\sigma T_2\right)=0, \quad (32)$$

$$2i\omega_3D_2A_3+\mu_3\left(i\omega_3\tau_3\right)^{\gamma}A_3+$$
$$+\left[\zeta_{13}\left(k_1+2k_2\right)+\zeta_{23}\left(k_3+2k_4\right)\right]A_3^2\bar{A}_3+$$
$$+\zeta_{13}\left(k_5+k_7\right)A_1\bar{A}_1A_3+\zeta_{23}\left(k_6+k_8\right)A_2\bar{A}_2A_3+ \quad (33)$$
$$+\left(\zeta_{13}k_8+\zeta_{23}k_5\right)\bar{A}_1A_2\bar{A}_3\exp\left(-2i\sigma T_2\right)-$$
$$-2f\exp\left(i\sigma_FT_2\right)=0.$$

Applying to (29)-(31) the same procedure as it has been done above for (11)–(13), as a result, we have

$$\left(a_1^2\right)^{\cdot}+s_1a_1^2 = 2\omega_1^{-1}\zeta_1k_8a_1a_2a_3^2\sin\delta, \quad (34)$$

$$\left(a_2^2\right)^{\cdot}+s_2a_2^2 = -2\omega_2^{-1}\zeta_2k_5a_1a_2a_3^2\sin\delta, \quad (35)$$

$$\left(a_3^2\right)^{\cdot}+s_3a_3^2 = -2f\omega_3^{-1}a_3\sin\beta_a+ \quad (36)$$
$$+\omega_3^{-1}\left(\zeta_{13}k_8+\zeta_{23}k_5\right)a_1a_2a_3^2\sin\delta,$$

$$\dot{\varphi}_1 = \frac{1}{2}\sigma_1+\omega_1^{-1}\zeta_1\left(k_5+k_7\right)a_3^2+ \quad (37)$$
$$+\omega_1^{-1}\zeta_1k_8a_1^{-1}a_2a_3^2\cos\delta,$$

$$\dot{\varphi}_2 = \frac{1}{2}\lambda_2 + \omega_2^{-1}\zeta_2\left(k_6 + k_8\right)a_3^2 + \\ +\omega_2^{-1}\zeta_2 k_5 a_1 a_2^{-1} a_3^2 \cos\delta, \tag{38}$$

$$\dot{\varphi}_3 = \frac{1}{2}\lambda_3 + \frac{1}{2}\omega_3^{-1}\zeta_{13}\left(k_5 + k_7\right)a_1^2 + \\ +\frac{1}{2}\omega_3^{-1}\zeta_{23}\left(k_6 + k_8\right)a_2^2 + \sigma_1 + \\ +\frac{1}{2}\omega_3^{-1}\left[\zeta_{13}\left(k_1 + 2k_2\right) + \zeta_{23}\left(k_3 + 2k_4\right)\right]a_3^2 + \\ +\frac{1}{2}\omega_3^{-1}\left(\zeta_{13}k_8 + \zeta_{23}k_5\right)a_1 a_2 \cos\delta - \\ -f\left(\omega_3 a_3\right)^{-1}\cos\beta_a, \tag{39}$$

where $\delta = 2\varphi_3 + \varphi_1 - \varphi_2$

is the phase difference.

The set of Eqs. (33)-(38) describes the phase-amplitude modulations at nonlinear forced vibrations (1)-(3) in the case of the difference combinational resonance (10).

3. NUMERICAL CALCULATIONS

The differential equations (15)-(20), (24)-(29) and (33)-(38) describing the phase-amplitude modulation for the additive and difference combinational resonances (8)-(10) have been solved numerically using the Runge-Kutta fourth-order algorithm at different magnitudes of the fractional parameter γ. The geometrical parameters of the plate utilized for calculations are given in Table 1 for three types of the combinational resonance for three types of plates: square, rectangular and oblong.

The envelopes of the amplitudes for all nine examples presented in Table 1 are shown in Figures 1-9 for free ($f = 0$) and forced ($f \neq 0$) vibrations, wherein solid, dotted and dashed lines correspond to the functions $a_3\left(T_2\right)$, $a_2\left(T_2\right)$ and $a_1\left(T_2\right)$, respectively, allowing one to trace the energy exchange between three interacting modes coupled by the additive-difference combinational resonances (8)-(10).

The time T_2-dependence of the amplitude envelopes for a rectangular plate with the dimensions a = 0.57 m and b = 0.1425 m (cases № 1, 4 and 7 in Table 1) are shown in Figures 1, 4 and 7 at $f = 10$ for three types of the combinational resonance. It is seen that the most unfavorable is the difference combinational resonance

$$2\omega_3 \simeq \omega_2 - \omega_1,$$

Table 1. Plate parameters which satisfy the combinational resonance condition.

№	ω_1	m_1	n_1	ω_2	m_2	n_2	ω_3	m_3	n_3	a, m	b, m	h, m	ν
Combinational resonance: $2\omega_3 \simeq \omega_1 + \omega_2$ and force amplitude level $f = 10$													
1	15.7	3	1	9.29	3	1	13.33	6	1	0.57	0.1425	0.0513	0.3
2	37.83	9	1	44.953	3	3	41.668	3	3	1.14	0.1425	0.0285	0.3
3	16.92	5	2	11.9	4	5	14.25	4	3	0.25	0.25	0.05	0.3
Combinational resonance: $2\omega_3 \simeq \omega_1 - \omega_2$ and force amplitude level $f = 10$													
4	42.15	6	3	14.985	1	2	13.324	6	1	0.57	0.1425	0.0513	0.3
5	100.58	1	4	14.985	1	1	41.65	3	3	1.14	0.1425	0.0285	0.3
6	32.345	9	5	4.156	1	2	14.245	4	3	0.25	0.25	0.05	0.3
Combinational resonance: $2\omega_3 \simeq \omega_2 - \omega_1$ and force amplitude level $f = 1$													
7	26.842	3	2	47.639	9	6	10.51	5	1	0.57	0.1425	0.0513	0.3
8	25.328	1	1	61.783	9	4	18.3	1	2	1.14	0.1425	0.0285	0.3
9	7.025	1	1	13.142	5	5	2.85	2	1	0.25	0.25	0.05	0.3

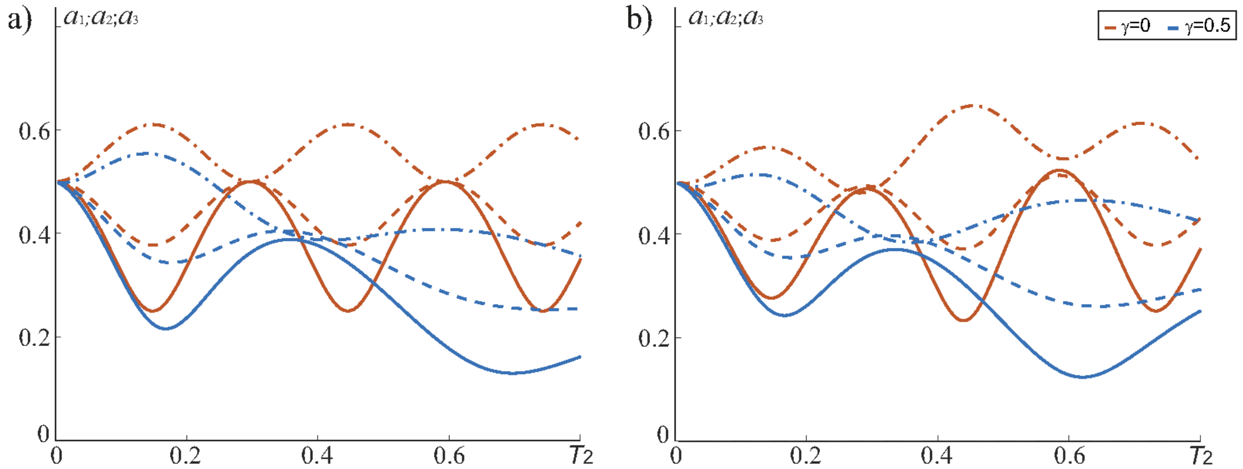

Figure 1. Amplitude envelopes of (a) free and (b) forced vibrations for plate №1 at the initial amplitudes $a_{i0} = 0.5$.

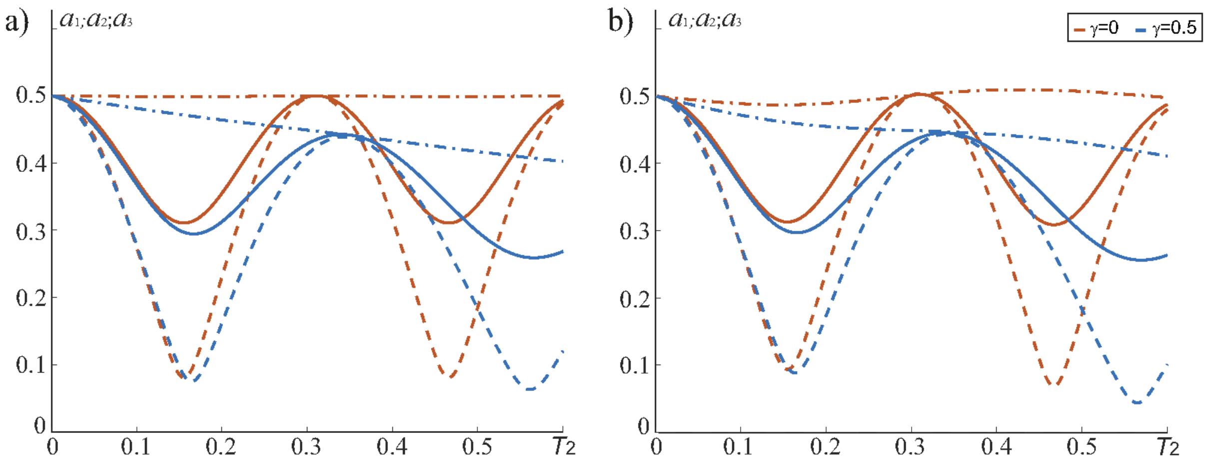

Figure 2. Amplitude envelopes of (a) free and (b) forced vibrations for plate №2 at $a_{i0} = 0.5$.

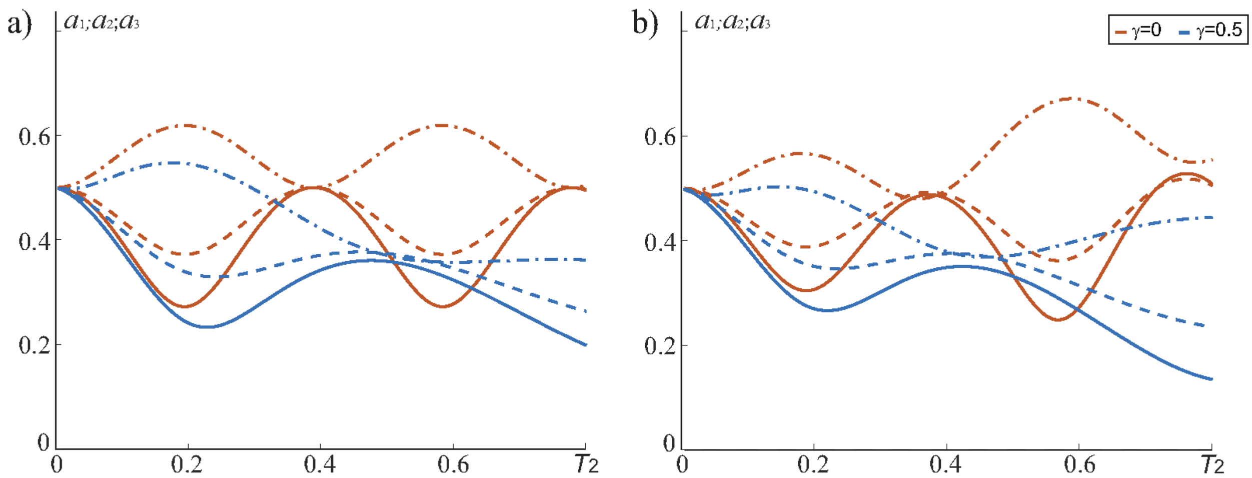

Figure 3. Amplitude envelopes of (a) free and (b) forced vibrations for plate №3 at $a_{i0} = 0.5$.

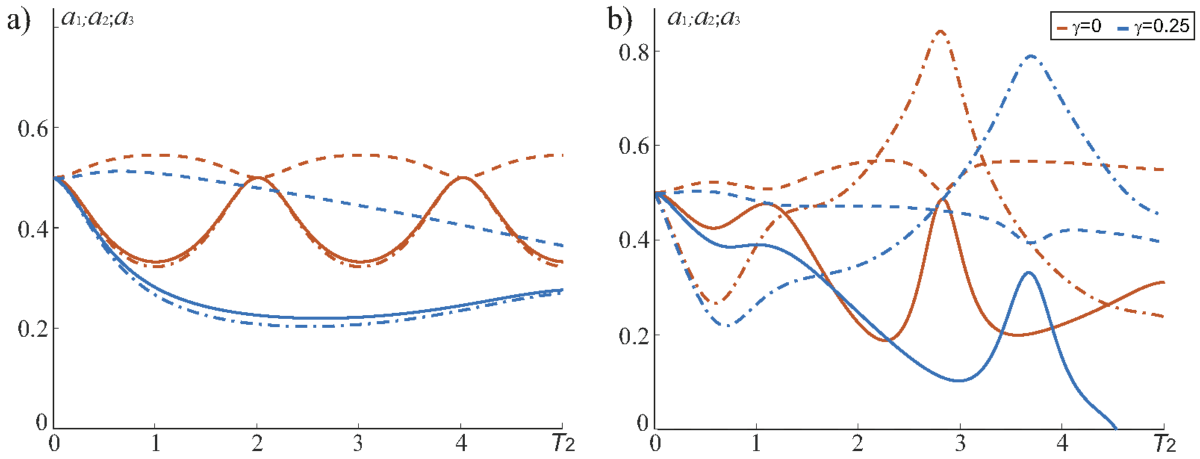

Figure 4. Amplitude envelopes of (a) free and (b) forced vibrations for plate №4 at $a_{i0} = 0.5$.

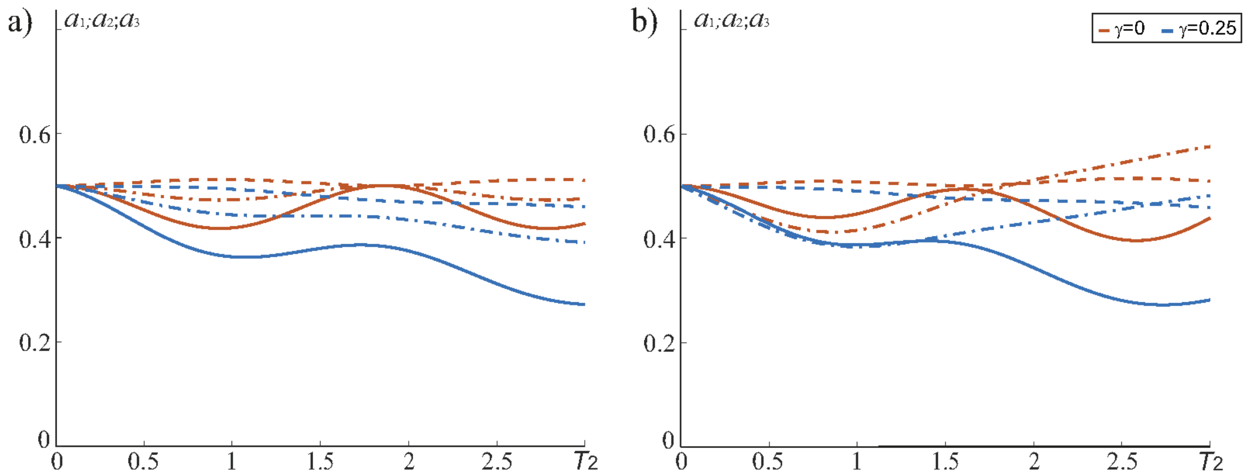

Figure 5. Amplitude envelopes of (a) free and (b) forced vibrations for plate №5 at $a_{i0} = 0.5$.

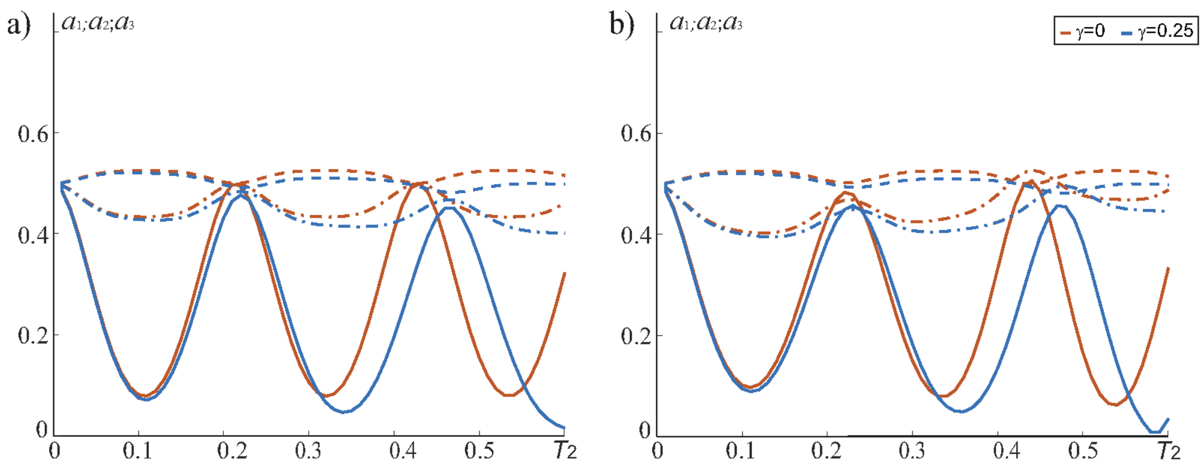

Figure 6. Amplitude envelopes of (a) free and (b) forced vibrations for plate №6 at $a_{i0} = 0.5$.

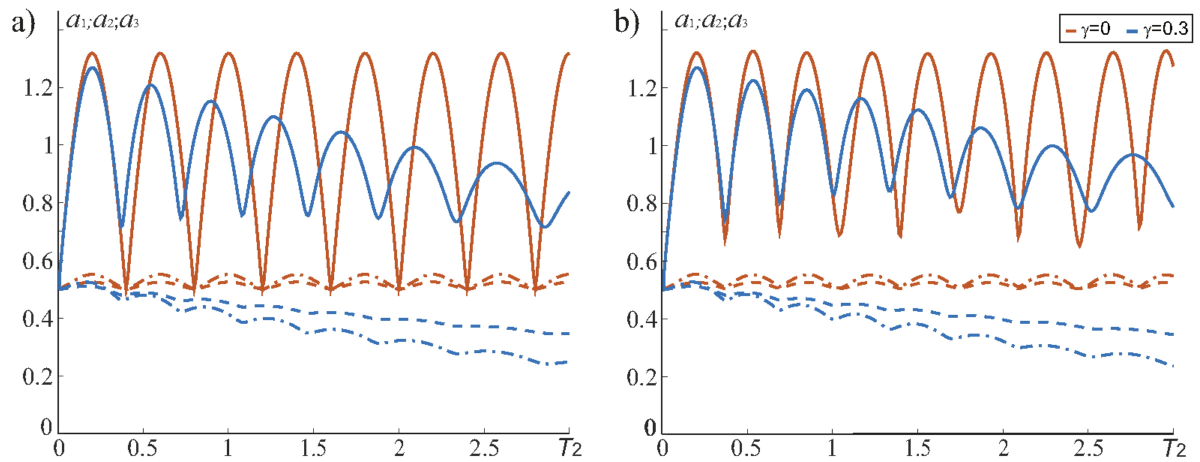

Figure 7. *Amplitude envelopes of (a) free and (b) forced vibrations for plate №7 at* $a_{i0} = 0.5$.

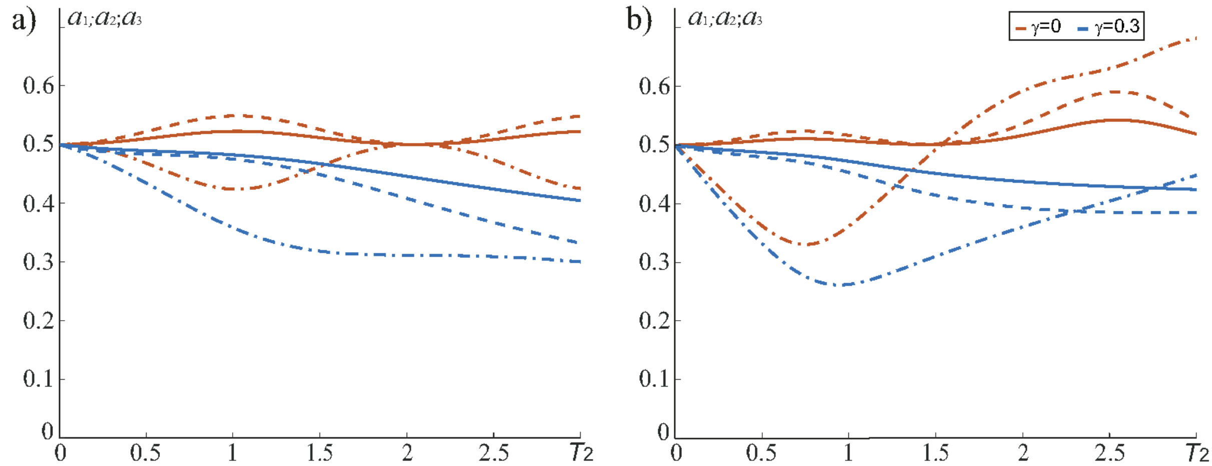

Figure 8. *Amplitude envelopes of (a) free and (b) forced vibrations for plate №8 at* $a_{i0} = 0.5$.

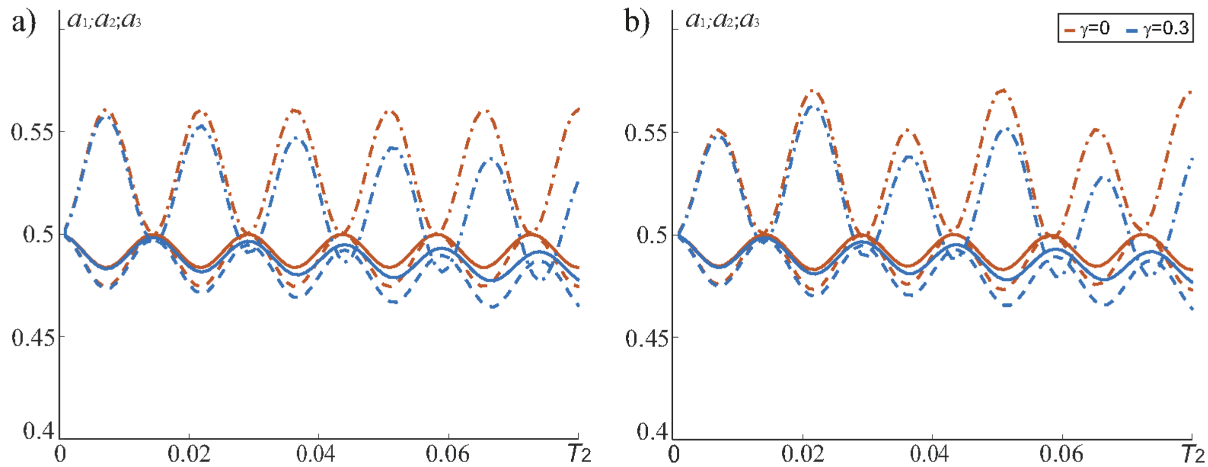

Figure 9. *Amplitude envelopes of (a) free and (b) forced vibrations for plate №9 at* $a_{i0} = 0.5$.

since it provides the essential increase in dimensionless amplitudes, resulting in high level of stresses and strains.

The time T_2-dependence of the amplitude envelopes for an oblong plate with dimensions *a*=1.14 m and *b*=0.1425 (cases 2, 5 and 8 in Table 1) is presented in Figures 2, 5 and 8 at $f = 10$, whence it follows that the additive combinational resonance is the most unfavorable, while the difference resonances result in the monotonic variation of dimensionless amplitudes.

As for a square plate (cases 3, 6 and 9 in Table 1), then all types of the combinational resonance influence equally on the amplitudes variation with time.

4. CONCLUSION

In the present paper, nonlinear force driven vibrations of thin plates in a viscoelastic medium have been studied, when the motion of the plate is described by a set of three coupled nonlinear differential equations subjected to the condition of the combinational resonance accompanied by the external resonance. Nonlinear sets of resolving equations in terms of amplitudes and phase differences have been solved numerically using the Runge-Kutta fourth-order algorithm. The influence of viscosity on the energy exchange mechanism between interacting modes has been analyzed. It has been revealed that plates of different dimensions behave in a different manner under the additive and difference combinational resonances. Rectangular plates are more sensitive to plate’s dimensions than square ones.

FUNDING

This research has been supported by the Ministry of Education and Science of the Russian Federation (Project 9.5138.2017/8.9).

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