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FORCED VIBRATIONS OF ANISOTROPIC ELASTIC SOLIDS SUBJECTED TO AN ACTION OF COMPLICATED LOADS

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Abstract: The work studies the forced vibrations of anisotropic elastic circular plates caused by dynamic loads uniformly distributed along concentric circumferences and over ring surfaces. The method of compensating loads (MCL) is used to solve the formulated problems. A new technique is used to construct basic and compensating solutions. The Nielsen's equation is taken into consideration. The solutions are obtained in closed form in terms of Bessel functions. Formulae of addition of cylindrical functions are used.

Key words: anisotropy, circular plates, forced vibrations, Bessel functions

ВЫНУЖДЕННЫЕ КОЛЕБАНИЯ АНИЗОТРОПНЫХ УПРУГИХ ТЕЛ ПРИ СЛОЖНЫХ НАГРУЗКАХ

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Аннотация: В работе изучаются вынужденные колебания анизотропных упругих круглых пластин, вызванные действием нагрузок, распределенных вдоль концентрических окружностей и по площадям колец. Для решения поставленных задач используется метод компенсирующих нагрузок (МКН). Для построения базовых решений – основного и компенсирующего, используется новый прием, берется в рассмотрение уравнение Нильсена. Решения получены в замкнутом виде и выражены в функциях Бесселя. Используются формулы сложения цилиндрических функций.

Ключевые слова: анизотропия, круглые пластины, вынужденные колебания, функции Бесселя

1. INTRODUCTION

In literature, for example in [1], the questions concerning analytical methods application to the solution of problems of statics and oscillations of elastic isotropic circular plates are covered. However, if the mentioned constructions are made from anisotropic material the application of analytical methods in the same ways as for isotropic ones causes considerable difficulties. For the cases of isotropic plates the resolving differential equation of the fourth order with variable coefficients is decomposed into two conjugate differential equations of the second order. The solutions are expressed in terms of Bessel functions. When solving the problems of anisotropic plates, the corresponding differential equation of the fourth order with variable coefficients does not decompose into two conjugate equations for any parameter values. This work applies the new approach for solving certain dynamic problems.
Nielsen's equation is used. First the similar approach was proposed for consideration of static problems of anisotropic circular plates resting on Winkler's foundation [2].
Let us assume that the plate material has cylindrical anisotropy and is orthotropic. The construction under study is subjected to an action of dynamic loads uniformly distributed along the lengths of concentric circles and over ring surfaces.

2. THE RESOLVING EQUATION

We present a differential equation describing the natural axisymmetric oscillation of a circular orthotropic plate [3], [4]:

$$r^4 \frac{\partial^4 w}{\partial r^4} + 2r^3 \frac{\partial^3 w}{\partial r^3} - n^2 r^2 \frac{\partial^2 w}{\partial r^2} + n^2 r \frac{\partial w}{\partial r} + \\ + \frac{\gamma h}{g n_2 D} \frac{\partial^2 w}{\partial t^2} = 0, \quad (1)$$

where D – cylindrical rigidity,

$$E_r = \frac{E}{n_1}, \; E_\theta = E n_2, \; \sigma_r = \frac{\sigma}{n^2}, \; \sigma_\theta = \sigma,$$

$$n^2 = n_1 n_2 .$$

When considering axisymmetric vibrations the solution of the equation (1) has the form:

$$w = \sum_{S=0}^{\infty} (A_S \cos p_S t + B_S \sin p_S t) W_S , \quad (2)$$

here p_S - circular frequency of natural vibrations, W_S - a function of only the coordinate r, constants A_S and B_S are determined from initial conditions, S is the number of nodal diameters.

As it was mentioned above, the studies have shown that for a plate made from orthotropic material, the initial fourth-order differential equation for any parameter values does not decay into two mutually adjoint second order equations as in the case for isotropic plates. Therefore, here to obtain an accurate analytical solution in terms of Bessel functions and for the application of the method of compensating loads (MCL) [2], [3], another technique was used. Nielsen’s equation was introduced into consideration. As a result the following solutions were obtained.

When the parameter $\mu = 0$, the general solution of the differential equation (2) has the form:

$$W_S = C_1 J_0(br) + C_2 Y_0(br) + C_3 I_0(br) + \\ + C_4 K_0(br); \quad (3)$$

for the parameter values $\mu = \pm 2$ the general solution of (2) is determined from the expression:

$$W_S = B_1 u_\mu(br) + B_2 \upsilon_\mu(br) + B_3 f_\mu(br) + \\ + B_4 g_\mu(br), \quad (4)$$

where

$$b = \sqrt[4]{\frac{\gamma h}{g n_2 D} p_S^2} .$$

3. THE METHOD OF COMPENSATING LOADS

In [3], forced oscillations of a circular orthotropic plate subjected to an action of a concentrated force in the center were studied. MCL was used for this purpose.

Below we will consider the forced vibrations of such plates under the action of much more complicated loads distributed along circumferences that do not coincide with the contour and loads distributed over areas of rings. MCL will also be used to build the solution. We introduce a dimensionless coordinate $x = br$. Let us assume that the plate under study is clamped along the contour.

First we will examine an action on the plate of the concentrated force $P \sin pt$ applied in the center.

Then the basic solution, which should contain a feature of the type of concentrated force, should be written in the form:

$$W_0 = C_0 Y_0(x) + D_0 K_0(x). \quad (5)$$

The compensating solution is determined by the formula:

$$W_k = A_0 J_0(x) + B_0 I_0(x). \quad (6)$$

In the expression (5) the functions $Y_0(x)$ and $K_0(x)$ are tend to infinity as $\ln x$. On the basis of the theory of Bessel functions we put

$$D_0 = \frac{2}{\pi} C_0.$$

The coefficient C_0 is determined from equilibrium conditions. Let us draw a circle of small radius x/b with the center in the point $x = 0$. We calculate the sum of the transverse forces acting on the circle of the mentioned radius and passage to the limit when $x \to 0$.
Performing all calculations [3], we get:

$$C_0 = -\frac{P}{8Dn_2b^2}. \tag{7}$$

As a result, (5) will take the form:

$$W_0 = -\frac{P}{8Dn_2b^2}\left[Y_0(x) + \frac{2}{\pi}K_0(x)\right]. \tag{8}$$

Summing up the basic and the compensating solutions, we define:

$$W = A_0 J_0(x) + B_0 I_0(x) - \frac{P}{8Dn_2b^2} \times \left[Y_0(x) + \frac{2}{\pi}K_0(x)\right]. \tag{9}$$

The constants A_0 and B_0 are determined from boundary conditions. We set in our case that when $x = \beta$ the outer boundary is clamped. When

$$x = \beta,\ w = 0,\ \frac{dw}{dx} = 0.$$

As a result of a number of transformations and using certain dependencies from Bessel's functions theory, we obtain the following expressions for A_0 and B_0:

$$A_0 = \frac{P}{8D_0n_2b^2} \times \frac{I_1(\beta)Y_0(\beta) + I_0(\beta)Y_1(\beta) + \frac{2}{\pi\beta}}{J_0(\beta)I_1(\beta) + J_1(\beta)I_0(\beta)},$$
$$B_0 = \frac{P}{8D_0n_2b^2} \times \frac{\frac{2}{\pi}[J_1(\beta)K_0(\beta) - J_0(\beta)K_1(\beta)] + \frac{2}{\pi\beta}}{J_0(\beta)I_1(\beta) + J_1(\beta)I_0(\beta)}. \tag{10}$$

In the case of another boundary conditions on the contour, the basic solutions will remain unchanged and the general form of the compensating solution will also remain. Due to the change of boundary conditions only the formulae for determining of the constants A_0 and B_0 will be different.

4. THE CALCULATION OF THE EFFECT OF LOAD DISTRIBUTED ALONG CIRCLES

We proceed to the study of the problem of forced oscillation of an anisotropic circular plate clamped along the contour caused by a load uniformly distributed around the circumference concentric with the contour and having a radius a_1. Denote the amplitude of the load $q \sin pt$ as q.
As it was shown above, we present the form of oscillation as the sum of the basic solution W_0 and the compensating solution W_k.
First we will define the basic solution. For this aim, mentally divide the loads acting along the circumference on the number of elementary loads. Next we sum-up the result of these loads. Take on the circle, on which the load is applied, a point with the coordinates (a_1, x). Elementary load acting on the section of arc with the length $a_1 d\theta$ is determined by the expression

$$\frac{qa_1 d\theta}{b}. \tag{11}$$

Find the deflection in the point with the reduced coordinates x, φ. To do this we use the formula (8). Substituting instead P the expression (11) and instead x the distance from the point of the elementary force application to the considered point of the plate, which is as follows:

$$z = \sqrt{a_1^2 + x^2 - 2a_1 x \cos(\theta - \varphi)}.$$

We make integration. Calculation of integrals should use the formulae of cylindrical functions addition, which in the studied case are of the form:

when $x \le a_1$

$$Y_0\left(\sqrt{a_1^2 + x^2 - 2a_1 x \cos(\theta - \varphi)}\right) = = 2\sum_0^{\infty}{}' J_n(x) Y_n(a_1) \cos n(\theta - \varphi), \tag{12}$$

$$K_0\left(\sqrt{a_1^2 + x^2 - 2a_1 x \cos(\theta - \varphi)}\right) = = 2\sum_0^{\infty}{}' I_n(x) K_n(a_1) \cos n(\theta - \varphi), \tag{13}$$

when $x \ge a_1$

$$Y_0\left(\sqrt{a_1^2 + x^2 - 2a_1 x \cos(\theta - \varphi)}\right) = = 2\sum_0^{\infty}{}' J_n(a_1) Y_n(x) \cos n(\theta - \varphi), \tag{14}$$

$$K_0\left(\sqrt{a_1^2 + x^2 - 2a_1 x \cos(\theta - \varphi)}\right) = = 2\sum_0^{\infty}{}' I_n(a_1) K_n(x) \cos n(\theta - \varphi). \tag{15}$$

In this formulae the sign ' means that for $n = 0$ an additional factor $1/2$ is introduced.
Write down:

$$w_0 = -\int_0^{2\pi} \frac{qa_1}{8Dn_2 b^3}\left[Y_0(z) + \frac{2}{\pi}K_0(z)\right] d\theta = = -\frac{qa_1}{8Dn_1 b^3} \times \times \left[\int_0^{2\pi} Y_0\left(\sqrt{a_1^2 + x^2 - 2a_1 x \cos(\theta - \varphi)}\right) d\theta + + \frac{2}{\pi}\int_0^{2\pi} K_0\left(\sqrt{a_1^2 + x^2 - 2a_1 x \cos(\theta - \varphi)}\right) d\theta\right]. \tag{16}$$

Substituting (12), (13), (14) and (15) in (16) and performing the integration; we note that all the series members when $n \ge 0$ at integration give zero. Therefore the solution will include only the result of integration of the zero term.
We have when $x \le a_1$

$$w_0 = -\frac{\pi q a_1}{4Dn_2 b^3} \times \times \left[J_0(x) Y_0(a_1) + \frac{2}{\pi} I_0(x) K_0(a_1)\right], \tag{17}$$

when $x > a_1$

$$w_0 = -\frac{\pi q a_1}{4Dn_2 b^3} \times \times \left[J_0(a_1) Y_0(x) + \frac{2}{\pi} I_0(a_1) K_0(x)\right]. \tag{18}$$

We receive the solution of the problem adding to the basic solution the compensating one:

$$w = w_0 + A_0 J_0(x) + B_0 I_0(x). \tag{19}$$

The coefficients A_0 and B_0 are determined from the boundary conditions. Here it was assumed that the outer contour of the plate when $x = \beta$ is clamped. Determining the mentioned coefficients A_0 and B_0 and using the expression for Wronskian, we get:

$$A_0 = \frac{\pi q a_1}{4Dn_2 b^3} \frac{J_0(a_1)[I_1(\beta)Y_0(\beta) + I_0(\beta)Y_1(\beta)] + \dfrac{2}{\pi\beta} I_0(a_1)}{I_0(\beta)J_1(\beta) + I_1(\beta)J_0(\beta)}, \quad (20)$$

$$B_0 = \frac{\pi q a_1}{4Dn_2 b^3} \frac{\dfrac{1}{\beta} J_0(a_1) + I_0(a_1)[I_1(\beta)K_0(\beta) - J_0(\beta)K_1(\beta)]}{I_0(\beta)J_1(\beta) + I_1(\beta)J_0(\beta)}. \quad (21)$$

For the cases of other boundary conditions only the equations for determining the constants A_0 and B_0 in the corresponding compensating solution will change.

5. THE LOADS DISTRIBUTED OVER RING SURFACES

We proceed to the study of forced oscillations caused by the load $q_1 \sin pt$, uniformly distributed over the area of the ring with the inner radius a_2 and the outer one a_3 $(a_2 < a_1 < a_3)$.
Let's define the basic solution. To do this the specified ring is divided into elementary concentric rings. The reduced inner radius of the elementary ring is denote by a_1, the outer one as $a_1 + da_1$. The load acting on the unit length of the ring is equal to

$$q = q_1 \frac{da_1}{b}.$$

The solution corresponding to the action of the elementary load is determined by the expressions (17) and (18) in which q should be replaced by

$$q_1 \frac{da_1}{b}.$$

Integrating these expressions in the range a_1 from a_2 to a_3, we obtain a solution of the problem under study. If the cross-section is in the inner circle when $x \le a_2$ which is free from external load, the expression (17) is to be integrated, that is

$$w_0 = -\frac{\pi q_1}{4Db^4} \times \\ \times \int_{a_2}^{a_3} \left[J_0(x)Y_0(a_1) + \frac{2}{\pi} I_0(x)K_0(a_1) \right] a_1 da_1. \quad (22)$$

From the theory of Bessel functions, for example from [1], the following relations are known:

$$\frac{d}{dx}\left[x^n Y_n(x)\right] = x^n Y_{n-1}(x), \quad (23)$$

$$\frac{d}{dx}\left[x^n K_n(x)\right] = -x^n K_{n-1}(x). \quad (24)$$

Assuming $n = 1$ we get:

$$\int xY_0(x)dx = xY_1(x), \quad (25)$$

$$\int xK_0(x)dx = -xK_1(x). \quad (26)$$

Taking into account (25) and (26) the expression for w_0 when $x \le a_2$ will take the following form:

$$w_0 = -\frac{\pi q_1}{4Dn_2 b^4} \{[a_3 Y_1(a_3) - a_2 Y_1(a_2)]J_0(x) - \\ -\frac{2}{\pi}[a_3 K_1(a_3) - a_2 K_1(a_2)]I_0(x)\}. \quad (27)$$

When considering a part $x \ge a_3$ the expression (18) must be integrated. To do this we should take into account the ratio [1]:

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x), \quad (28)$$

$$\frac{d}{dx}\left[x^n I_n(x)\right] = x^n I_{n-1}(x); \quad (29)$$

hence for $n=1$ we get:

$$\int xJ_0(x)dx = xJ_1(x), \quad (30)$$

$$\int xI_0(x)dx = xI_1(x). \quad (31)$$

Thus when $x \ge a_3$ we get:

$$w_0 = -\frac{\pi q_1}{4Dn_2b^4}\{[a_3J_1(a_3) - a_2J_1(a_2)]Y_0(x) + \\ + \frac{2}{\pi}[a_3I_1(a_3) - a_2I_1(a_2)]K_0(x)\}. \quad (32)$$

When considering the area under the load when $a_2 \le x \le a_3$ to determine the solution of the problem we should integrate (18) when $a_2 \le a_1 \le x$ and when $a_1 \le x \le a_3$ we should integrate (17), replacing q on

$$q_1\frac{da_1}{b}$$

as elsewhere. At the same time, using the formulae (25), (26), (30) and (31), we get the following result when $a_2 \le x \le a_1$:

$$w_0 = -\frac{\pi q_1}{4Dn_2b^4}\{[xJ_1(x) - a_3J_1(a_2)] - \\ -Y_0(x) + \frac{2}{\pi}[xI_1(x) - a_2I_1(a_2)]K_0(x) + \\ + [a_3Y_1(a_3) - xY_1(x)]J_0(x) - \\ - \frac{2}{\pi}[a_3K_1(a_3) - xK_1(x)]I_0(x)\}. \quad (33)$$

By regrouping we obtain:

$$w_0 = -\frac{\pi q_1}{4Dn_2b^4}\{x[J_1(x)Y_0(x) - Y_1(x)J_0(x)] + \\ + \frac{2x}{\pi}[I_1(x)K_0(x) + I_0(x)K_1(x)] - \\ -a_2J_1(a_2)Y_0(x) - \frac{2}{\pi}a_2I_1(a_2)K_0(x) + \\ + a_3Y_1(a_3)J_0(x) - \frac{2}{\pi}a_3K_1(a_3)I_0(x)\}. \quad (34)$$

Using the expression for Wronskian we get:

$$w_0 = -\frac{\pi q_1}{4Dn_2b^4}\left[\frac{4}{\pi} - a_2J_1(a_2)Y_0(x) - \\ - \frac{2}{\pi}a_2I_1(a_2)K_0(x) + a_3Y_1(a_3)J_0(x) - \\ - \frac{2}{\pi}a_3K_1(a_3)I_0(x)\right]. \quad (35)$$

Summing the basic and compensating solutions we obtain the expression which is similar to (19) where the constants A_0 and B_0 are determined from the boundary conditions.

6. CONCLUSION

The present work for the first time receives the exact analytical solutions of the problems of forced vibrations of circular plates which are made from material having cylindrical anisotropy. The constructions under study are subjected to an action of dynamic loads uniformly distributed along the lengths of concentric circumferences and over areas of ring surfaces. Method of compensating loads for determination of the solutions is used. The solutions are obtained in terms of Bessel functions.

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