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MATHEMATIC MODEL OF A BEAM PARTIALLY SUPPORTED ON ELASTIC FOUNDATION

***Vladimir I. Travush* [1], *Vladimir A. Gordom* [2], *Vitaly I. Kolchunov* [3], *Yevgeny V. Leontiev* [4]**

[1] Russian Academy of Architecture and Construction Sciences, Moscow, RUSSIA
[2] Orel State University named after I.S. Turgenev, Orel, RUSSIA
[3] Southwest State University, Kursk, RUSSIA
[4] Main Department of State Expertise, Moscow, RUSSIA

Abstract: The paper presents the methodic for analytical determining of stress-strain state of a beam partially supported on elastic foundation at sudden damage of foundation structure (partial failure). Bending equation for a beam is written using dimensional parameter and solved with the initial parameters method. Such approach allows to obtain dimensional analytical solution to static and dynamic problems for universal boundary conditions of a beam since it always leads to equations' system of second order. Using numerical analysis for various values of generalized stiffness parameter of a system "beam – foundation", we established affecting of the length of failure foundation part to stress-strain state of the beam for two supporting variants: partial supporting and supporting by two ends with foundation failure in the middle part of the beam.

Keywords: system "beam – foundation", accidental impact, sudden foundation failure, forces, deflections

МАТЕМАТИЧЕСКАЯ МОДЕЛЬ БАЛКИ, ЧАСТИЧНО ОПЕРТОЙ НА УПРУГОЕ ОСНОВАНИЕ

***В.И. Травуш* [1], *В.А. Гордон* [2], *В.И. Колчунов* [3], *Е.В. Леонтьев* [4]**

[1] Российская академия архитектуры и строительных наук, г. Москва, РОССИЯ
[2] Орловский государственный университет имени И.С. Тургенева, г. Орел, РОССИЯ
[3] Юго-западный Государственный университет, г. Курск, РОССИЯ
[4] Главгосэкспертиза России, г. Москва, РОССИЯ

Аннотация: Приведена методика аналитического определения усилий напряженно деформированного состояния балки частично опертой на упругое основание при внезапном дефекте основания – частичного ее разрушения. Уравнение изгиба балки записаны в безразмерных координатах и решены методов начальных параметров это позволяет получать безразмерные аналитические решения статических и динамических задач универсальными по отношению к условиям закрепления балки, поскольку всегда приводит к системе уравнений второго порядка. Численным анализам для различных значений параметра обобщенной жесткости системы «балка-основание» установлено влияние длинны утраченной части основания на напряженно деформированное состояние балки при двух вариантах опирания на упругое основание: частичное опирание и опирание по двухопорной схеме с разрушением основания в средней части пролета.

Ключевые слова: система «балка-основание», аварийное воздействие, внезапный дефект основания, усилия, перемещения

1. INTRODUCTION

Traditional approach for solving the problem of static and dynamic flexure of a beam supported on elastic foundation with discrete changes of geometric and (or) mechanical characteristics of this beam and (or) foundation along it length is representation of a system "beam – foundation" as series of rod sections supported on elastic foundation in the boundaries of which proper-

ties of the beam and foundation change it values continuously or constant [1-6]. At the same time, it leads to solving of Cauchy problem for system of ordinary differential equations with variables parameters. Besides boundary conditions, solving of these equations requires accepting continuity conditions for coefficients in the discontinuity points of the first order (conjunction of sections). For many sections, such approach is laborious since it leads to a large number N of equations for determining of integration constants ($N = 4n$, where n is sections' number). In this case, the initial parameters' method (IPM) becomes effective [7-10]. The method allows to obtain dimension less analytical solutions to static and dynamic problems. These solutions are universal for different types of boundary conditions and number of sections since it always leads to equations' system of second order. It is conveniently to apply the IPM procedure in addition with state vectors for an arbitrary cross section of a beam and matrixes of initial cross section affecting to arbitrary cross section [6, 8, 9, 11-14].

This paper presents the first part of investigation of dynamic interaction of a continuous footing with foundation under it. The continuous footing is simulated as an elastic flexible Bernoulli-Euler beam of finite length and elastic foundation under it. Despite a large number of literature to problems linked with rod systems supported on different types of foundation, there is not advanced analysis of the case, when statically loaded system during operation loss it foundation or its part. In according to Structural Mechanics, constructively nonlinear system appears un this case. This process causes appearing of additional dynamic forces in the system. While not all accidental impacts are classified and reactions of structural elements are not investigated enough. Since, the present study is actual, theoretically interesting and practically significant.

The first part of the problem for constructing and investigating of the mathematic models of dynamic transmitting processes into loaded beam at sudden foundation damage (partial failure) is analysis of stress-strain state of loaded beam at quasi-static failure of the supporting foundation part. The case of entire foundation failure is not considered because it deletes the subject of investigation.

Solutions to static problems are used further to formulate initial conditions for dynamic problems and for comparison of stress-strain state before and after appearing of a damage.

2. MATHEMATIC MODEL

2.1. Cantilever beam partially supported on Winkler foundation.

Beam of l length, of cross section area A and axial inertia moment I, made of material of ρdensity and with elasticity modulus E supports partially on elastic Winkler foundation of k stiffness (Figure 1.1).

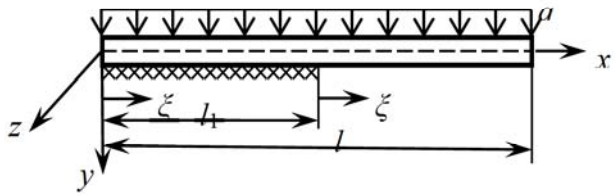

Figure 2.1. Calculational model for a beam partially supported on elastic foundation.

The length of the first section is l_1. Let us assume that beam's ends with coordinates $x = 0$ and $x = l$ are free. The beam is loaded along the span with evenly distributed load q. It is formulated the problem for determining of deflections and bending moments in dependence to length of the supported section. Beam is considered as a composite one after foundation failure and consists of two homogeneous sections supported on a step foundation of partially constant stiffness: $k_1 = k$ and $k_2 = 0$. We construct solution to this problem separately for each section using local coordinates x_i, y_i, z_i $(i = 1, 2)$: x_i is longitudinal axis, y_i and z_i is main central inertia axis of the cross section. When we chose the origin point at initial cross section of each section, the

continuity condition for solution to this problem means that state of the end cross section of previous section is the initial for the next one. Let us introduce dimensionless coordinates and parameters

$$\xi_i = \frac{x_i}{l},\ w_i = \frac{v_i}{l},\ (i = 1,2)$$

– deflection of the beam;

$$v = \frac{l_1}{l}$$

– relative length of the supported part of the beam;

$$\overline{q} = \frac{ql^3}{EI}$$

– external load;

$$\alpha = \sqrt[4]{\frac{kl^4}{4EI}}$$

– generalized stiffness parameter of a system "beam – foundation".

Consider the successive bending of each section.

2.2. Bending of the first section $0 \le \xi_1 \le v$.

Solution to bending equation

$$\frac{d^4 w_1}{d\xi_1^4} + 4\alpha w_1 = \overline{q}, \qquad (2.1)$$

expressed through the initial parameters of this section

$$w_{10} = w_1(0), w'_{10} = w'_1(0), w''_{10} = w''_1(0), w'''_{10} = w'''_1(0)$$

takes the form [6, 12, 13]

$$w_1(\xi_1) = \frac{\overline{q}}{4\alpha^4}(1 - K_4(\alpha\xi_1)) + w_{10}K_4(\alpha\xi_1) + \\ + w'_{10}K_3(\alpha\xi_1) + w''_{10}K_2(\alpha\xi_1) + w'''_{10}K_1(\alpha\xi_1),$$

where $K_i(\alpha\xi_1)$ is Krylov function [7, 14]:

$$K_1 = \frac{\sin\alpha\xi_1 ch\alpha\xi_1 - \cos\alpha\xi_1 sh\alpha\xi_1}{4\alpha^3}, K_2 = K'_1,$$
$$K_3 = K'_2, K_4 = K'_3, K_1 = \frac{1}{4\alpha 4}K'_4$$

State of an arbitrary cross section of the first segment of the beam can be expressed by matrix equation

$$\overline{w}_1 = V_1(\xi_1)\overline{W}_{10} + \overline{q}\,\overline{V}_{1q}(\xi_1), \qquad (2.2)$$

where $\overline{w}_1(\xi_1) = \{w_1(\xi_1)\ w'_1(\xi_1)\ w''_1(\xi_1)\ w'''_1(\xi_1)\}^T$ is state vector for an arbitrary cross section ξ_1 of the first section; $\overline{W}_{10} = \{w_{10}\ w'_{10}\ w''_{10}\ w'''_{10}\}^T$ – vector of initial parameters of the first section;

$$\underset{4\times4}{V_1(\xi_1)} = \begin{pmatrix} K_4(\alpha\xi_1) & K_3(\alpha\xi_1) & K_2(\alpha\xi_1) & K_1(\alpha\xi_1) \\ -4\alpha^4 K_1(\alpha\xi_1) & K_4(\alpha\xi_1) & K_3(\alpha\xi_1) & K_2(\alpha\xi_1) \\ -4\alpha^4 K_2(\alpha\xi_1) & -4\alpha^4 K_1(\alpha\xi_1) & K_4(\alpha\xi_1) & K_3(\alpha\xi_1) \\ -4\alpha^4 K_3(\alpha\xi_1) & -4\alpha^4 K_2(\alpha\xi_1) & -4\alpha^4 K_1(\alpha\xi_1) & K_4(\alpha\xi_1) \end{pmatrix}$$ – matrix of initial parameters of the first section affecting to state of section ξ_1 of this section;

$$\overline{V}_{1q}(\xi_1) = \left\{\frac{1 - K_4(\alpha\xi_1)}{4\alpha^4}\quad K_2(\alpha\xi_1)\quad K_2(\alpha\xi_1)\quad K_3(\alpha\xi_1)\right\}^T$$ – vector of forces for the first section.

2.3. Bending of the second section

$0 \le \xi_2 \le 1 - \nu$.

Flexure of this section can be expressed by equation

$$\frac{d^4 w_2}{d\xi_2^4} = \overline{q}, \tag{2.3}$$

general solution to which in the form of the initial parameters' method should be written as [6]

$$w_2(\xi_2) = \frac{\overline{q}\xi_2^4}{24} + w_{20} + w'_{20}\xi_2 + w''_{20}\frac{\xi_2^2}{2} + w'''_{20}\frac{\xi_2^3}{6}$$

or in the matrix form

$$\overline{w}_2(\xi_2) = V_2(\xi_2)\overline{W}_{20} + \overline{q}\,\overline{V}_{2q}, \tag{2.4}$$

where

$$\overline{w}_2(\xi_2) = \{w_2(\xi_2)\ w'_2(\xi_2)\ w''_2(\xi_2)\ w'''_2(\xi_2)\}^T$$

– state vector of an arbitrary cross section ξ_2 of the second section;

$$\overline{W}_{20} = \{w_{20}\ w'_{20}\ w''_{20}\ w'''_{20}\}^T$$

– vector of initial parameters for the second section;

$$\underset{4\times4}{V_2(\xi_2)} = \begin{pmatrix} 1 & \xi_2 & \frac{\xi_2^2}{2} & \frac{\xi_2^3}{6} \\ 0 & 1 & \xi_2 & \frac{\xi_2^2}{2} \\ 0 & 0 & 1 & \xi_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

– matrix of initial parameters for second section affecting to state of a cross section ξ_2 of this segment;

$$\overline{V}_{2q}(\xi_2) = \left\{\frac{\xi_2^4}{24}\quad \frac{\xi_2^3}{6}\quad \frac{\xi_2^2}{2}\quad \xi_2\right\}^T$$

– force vector for the second segment.

2.4. Conjunction condition for segments.

Conjunction condition of segments can be expressed in matrix form as equality of vectors

$$\overline{w}_1(\nu) = \overline{w}_2(0), \tag{2.5}$$

From equation (2.2) for $\xi_1 = \nu$ it follows

$$\overline{w}_1(\nu) = V_1(\nu)\overline{W}_{10} + \overline{q}\,\overline{V}_{1q}(\nu), \tag{2.6}$$

and from equation (2.4) for $\xi_2 = 0$, we obtain

$$\overline{w}_2(0) = \overline{W}_{20} \tag{2.7}$$

since the matrix $V_2(0)$ is singular, and vector $\overline{V}_{2q}(0) = \overline{0}$. Substituting (2.6) and (2.7) into (2.5), we obtain expression of initial parameters for the second segment through initial parameters of the first segment

$$\overline{W}_{20} = V_1(\nu)\overline{W}_{10} + \overline{q}\,\overline{V}_{1q}(\nu).$$

Then from equation (2.4), it follows

$$\begin{aligned}\overline{w}_2(\xi_2) &= V_{21}(\xi_2;\nu)\overline{W}_{10} + \\ &+ \overline{q}\left(\overline{V}_{21q}(\xi_2;\nu) + \overline{V}_{2q}(\xi_2)\right),\end{aligned} \tag{2.8}$$

where

$$\underset{4\times4}{V_{21}} = V_2(\xi_2)V_1(\nu)$$

– affecting matrix of initial parameters for the first segment to state of an arbitrary cross section ξ_2 of the second segment;

$$\overline{V}_{21q} = V_2(\xi_2)\overline{V}_{1q}(\nu)$$

– vector of additional loads for the second segment.

Thus, state of both segments of the beam is expressed through initial parameters of the first segment. Two parameters are known in advance from condition for the left end of the beam ($\xi_1 = 0$). The next two unknown parameters can be determined from conditions at the right end of the beam for

$$\xi_2 = 1 - \nu.$$

2.5. Determination of unknown initial parameters.

Let us write matrix equation (2.8) in the extended form

$$\begin{pmatrix} w_2(\xi_2) \\ w_2'(\xi_2) \\ w_2''(\xi_2) \\ w_2'''(\xi_2) \end{pmatrix} = \begin{pmatrix} 1 & \xi_2 & \frac{\xi_2^2}{2} & \frac{\xi_2^3}{6} \\ 0 & 1 & \xi_2 & \frac{\xi_2^2}{2} \\ 0 & 0 & 1 & \xi_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} K_4(\alpha\xi_1) & K_3(\alpha\xi_1) & K_2(\alpha\xi_1) & K_1(\alpha\xi_1) \\ -4\alpha^4 K_1(\alpha\xi_1) & K_4(\alpha\xi_1) & K_3(\alpha\xi_1) & K_2(\alpha\xi_1) \\ -4\alpha^4 K_2(\alpha\xi_1) & -4\alpha^4 K_1(\alpha\xi_1) & K_4(\alpha\xi_1) & K_3(\alpha\xi_1) \\ -4\alpha^4 K_3(\alpha\xi_1) & -4\alpha^4 K_2(\alpha\xi_1) & -4\alpha^4 K_1(\alpha\xi_1) & K_4(\alpha\xi_1) \end{pmatrix} \begin{pmatrix} w_{10} \\ w_{10}' \\ w_{10}'' \\ w_{10}''' \end{pmatrix} +$$

$$+\bar{q}\left[\begin{pmatrix} 1 & \xi_2 & \frac{\xi_2^2}{2} & \frac{\xi_2^3}{6} \\ 0 & 1 & \xi_2 & \frac{\xi_2^2}{2} \\ 0 & 0 & 1 & \xi_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1-K_4(\alpha\nu)}{4\alpha^4} \\ K_1(\alpha\nu) \\ K_2(\alpha\nu) \\ K_3(\alpha\nu) \end{pmatrix} + \begin{pmatrix} \frac{\xi_2^4}{24} \\ \frac{\xi_2^3}{6} \\ \frac{\xi_2^2}{2} \\ \xi_2 \end{pmatrix}\right] \tag{2.9}$$

In order to further construction, it is need to take in account restraints at the ends of the beam. Then boundary conditions take the form In order to further construction, it is need to take in account restraints at the ends of the beam. Then boundary conditions take the form

$$\begin{aligned} w_1''(0) = w_1'''(0) = 0 \\ w_2''(1-\nu) = w_2'''(1-\nu) = 0 \end{aligned} \tag{2.10}$$

From the first two conditions (2.10), it follows

$$w_{10}'' = w_{10}''' = 0.$$

From the second group of conditions (2.10) using equation (2.9), we obtain parameters w_{10} and w_{10}', as solutions to the system of algebraic equations

$$\begin{cases} (K_2(\alpha\nu) + (1-\nu)K_3(\alpha\nu))w_{10} + \\ + (K_1(\alpha\nu) + (1-\nu)K_2(\alpha\nu))w_{10}' = \\ = \frac{\bar{q}}{4\alpha^4}\left(K_2(\alpha\nu) + (1-\nu)K_3(\alpha\nu) + \left(\frac{(1-\nu)^2}{2}\right)\right) \\ K_3(\alpha\nu)w_{10} + K_2(\alpha\nu)w_{10}' = \\ = \frac{\bar{q}}{4\alpha^4}(K_3(\alpha\nu) + 1 - \nu) \end{cases}$$

From where we obtain

$$w_{10} = \frac{\bar{q}}{4\alpha^4}\left(1 - \frac{(1-\nu)K_1(\alpha\nu) + \frac{(1-\nu)^2}{2}K_2(\alpha\nu)}{K_2^2(\alpha\nu) - K_1(\alpha\nu)K_3(\alpha\nu)}\right)$$

$$w_{10}' = \frac{\bar{q}}{4\alpha^4} \cdot \frac{(1-\nu)K_2(\alpha\nu) + \frac{(1-\nu)^2}{2}K_3(\alpha\nu)}{K_2^2(\alpha\nu) - K_1(\alpha\nu)K_3(\alpha\nu)}$$

In accordance with matrix equation (2.2), deflections $w_1(\xi_1)$ and bending moments $w_1''(\xi_1)$ in an arbitrary section ξ_1 of the first segment are determined by functions

$$w_1(\xi_1) = K_4(\alpha\xi_1)w_{10} + K_3(\alpha\xi_1)w'_{10} + \\ + \frac{\bar{q}}{4\alpha^4}(1 - K_4(\alpha\xi_1)), \tag{2.11}$$

$$w_1(\xi_1) = -4\alpha^4(K_2(\alpha\xi_1)w_{10} + \\ + K_1(\alpha\xi_1)w'_{10}) + \bar{q}\,K_2(\alpha\xi_1). \tag{2.12}$$

Deflections $w_2(\xi_2)$ and bending moments $w_2''(\xi_2)$ in an arbitrary cross section ξ_2 $w_2(\xi_2)$ of the second segment are determined by functions from matrix equation (2.8)

$$w_2(\xi_2) = C_0 + C_1\xi_2 + C_2\xi_2^2 + \\ + C_3\xi_2^3 + C_4\xi_2^4, \tag{2.13}$$

$$w_2''(\xi_2) = 2C_2 + 6C_3\xi_2 + 12C_4\xi_2^2, \tag{2.14}$$

where

$$C_0 = K_4(\alpha v)w_{10} + K_3(\alpha v)w'_{10} + \frac{\bar{q}}{4\alpha^4}(1 - K_4(\alpha v));$$

$$C_1 = -4\alpha^4 K_1(\alpha v)w_{10} + K_4(\alpha v)w'_{10} + \bar{q}\,K_1(\alpha v);$$

$$C_2 = -2\alpha^4 K_2(\alpha v)w_{10} - 2\alpha^4 K_1(\alpha v)w'_{10} + \\ + \frac{\bar{q}}{2}K_2(\alpha v);$$

$$C_3 = -\frac{2}{3}\alpha^4 K_3(\alpha v)w_{10} - \frac{2}{3}\alpha^4 K_2(\alpha v)w'_{10} + \\ + \frac{\bar{q}}{6}K_3(\alpha v);$$

$$C_4 = \frac{\bar{q}}{24}.$$

2.6. Numerical results.

Using Maple we carried out calculation of dimension less deflections $w(\xi)$ and bending moments $w''(\xi)$ for a beam, partially supported on elastic foundation and loaded with evenly distributed load $\bar{q} = 1$. Calculations are performed for different values of generalized stiffness parameter α of a system "beam – foundation" in order to determine influence of damage value (length of failure part of foundation) to stress-strain state of the beam at increasing such damage from one of end cross sections of this beam. Figures 2.2-2.6 and Table 2.1 present such calculation results.

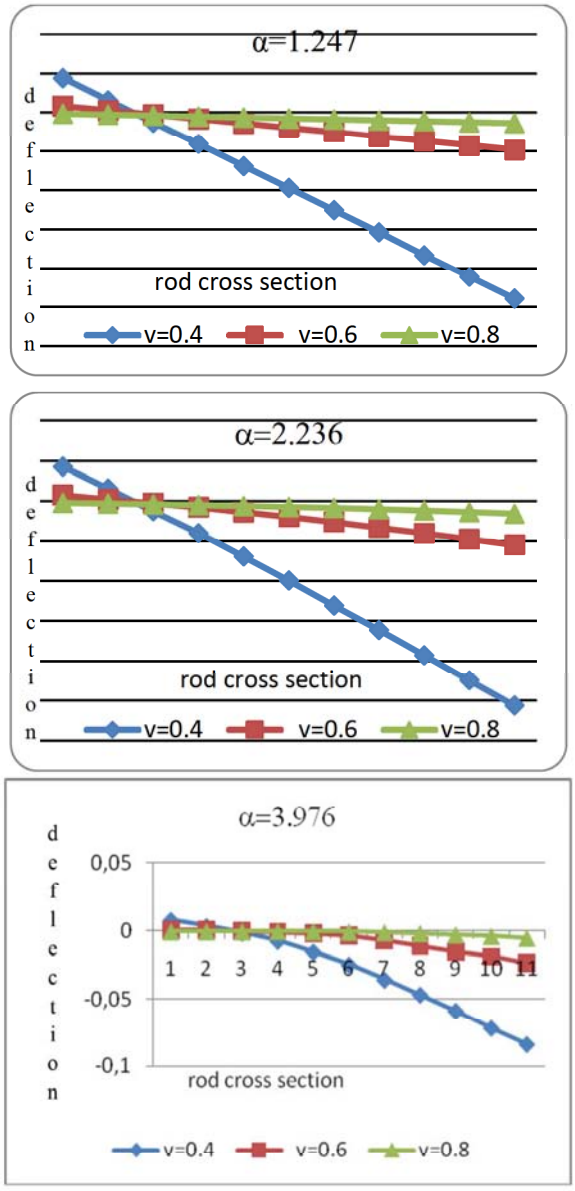

Figure 2.2. Diagrams for comparison deflections of the beam at length decreasing of supported part of the beam.

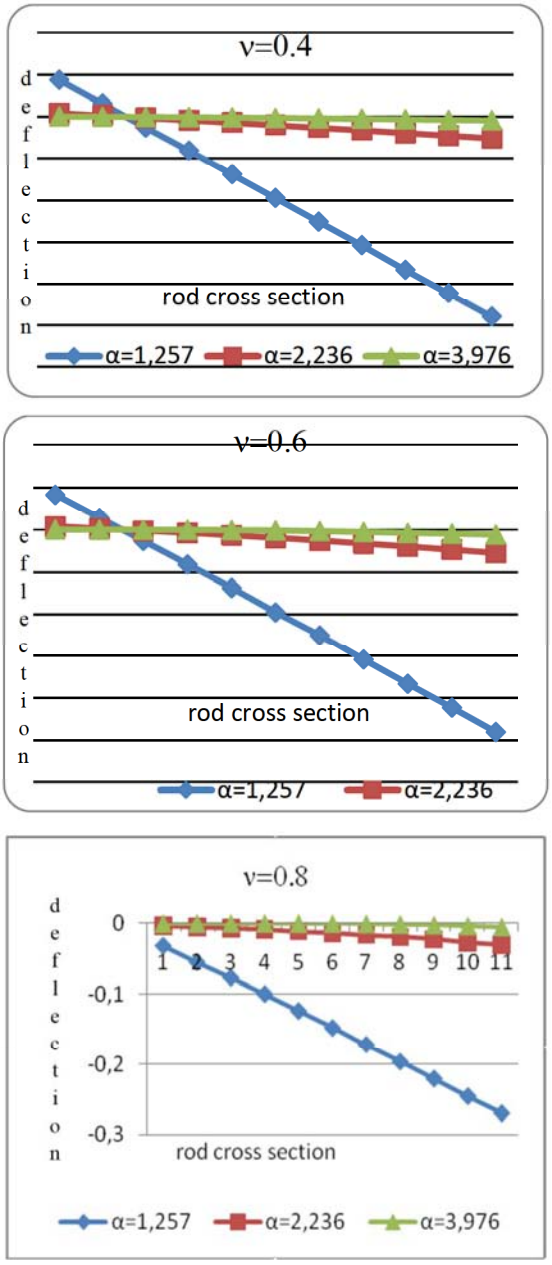

Figure 2.3. Diagrams for comparison of deflections at constant damage sizes.

Figure 2.2 shows diagrams for comparison deflections of the beam when supported part of the beam decreases for values $\nu = 0{,}8; 0{,}6; 0{,}4$ (or, respectively, damage length increases for values $1-\nu = 0{,}2; 0{,}4; 0{,}6$) for three values of α parameter of generalized stiffness of a system "beam - foundation": $\alpha = 1{,}275; 2{,}236; 3{,}976$.

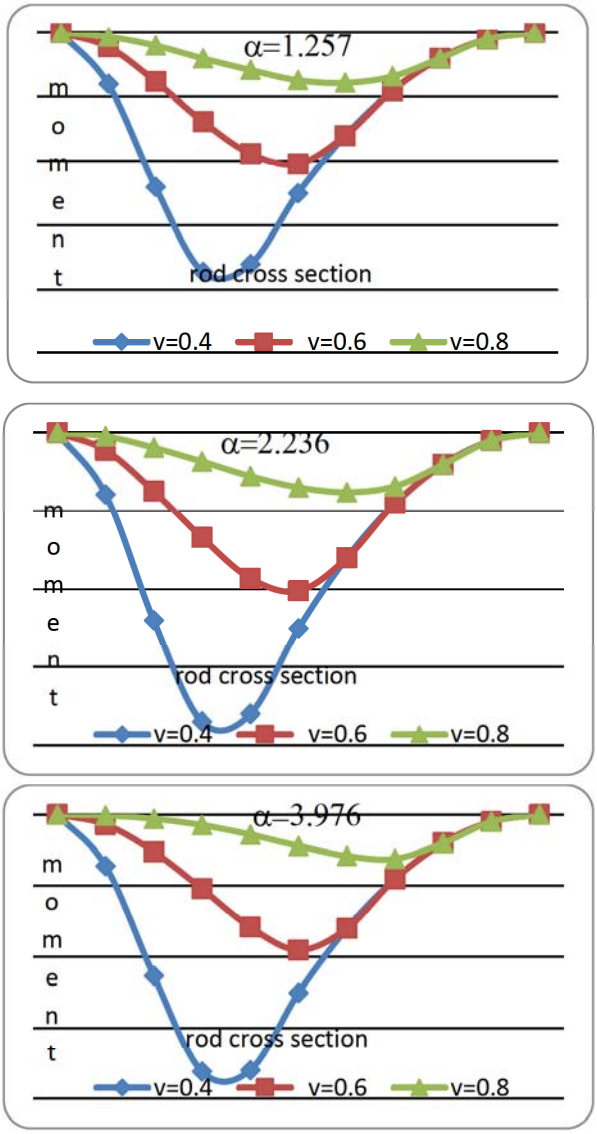

Figure 2.4. Diagrams of bending moments for different combinations of stiffness variants α.

Figure 2.3 shows comparison of diagrams of deflections at constant values of damage length $1-\nu = 0{,}2; 0{,}4; 0{,}6$ for different generalized stiffness of a system "beam – foundation": $\alpha = 1{,}275; 2{,}236; 3{,}976$. Let us note that in accordance with Winkler model, a free beam entirely supported on foundation ($\nu = 1$) and loaded with evenly distributed load $\bar{q}$, moves in parallel itself without flexure for value $\bar{q}/4\alpha^4$.

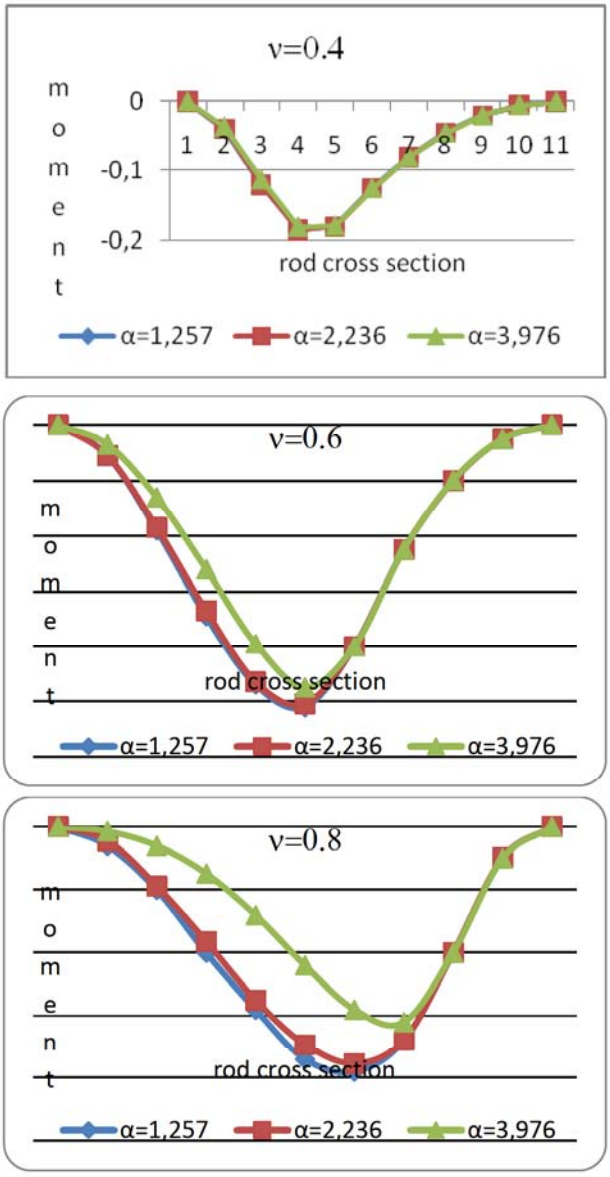

Figure 2.5. Diagrams of bending moments for different combinations of damage sizes $(1-\nu)$.

Diagrams' character for all variants of system stiffness α and damage sizes ν is similar: maximum deflections increase with increasing length of cantilever part for constant system stiffness, and it decreases for large values of stiffness for constant length of cantilever.

Figures 2.4 and 2.5 presents diagrams of bending moments for different combinations of stiffness α and damage values $1-\nu$.

The maximum moment for all cases arises in the supported part of the beam and moves to the left end of the beam at damage increasing. Value of maximum moment for all stiffness variants α significantly depends on the damage value

$$(1-\nu): w_{\max\limits_{1-\nu=0,6}} \geq w_{\max\limits_{1-\nu=0,4}} \geq w_{\max\limits_{1-\nu=0,2}}$$

(Figure 2.4) and does not depend on system stiffness α for constant length of damaged part $(1-\nu)$ (Figure 2.5).

Table 2.1 presents values of maximum deflections and bending moments for considered stiffness variants of a system "beam – foundation" and sizes of supported part of the beam ν.

Let us present an example of a system "beam – foundation", the generalized stiffness of which takes the value $\alpha = 3{,}976$. It is reinforced concrete beam of $l = 6.7$ m length with rectangular cross section and sizes $b = 0.25$ m (width), $h = 0.18$ m (height), $A = 0.045$ m^2 is area of cross section. Inertia moment of cross section is $I = 1{,}215 \cdot 10^{-4}$ m^4. Elasticity modulus for beam material is $E = 3.05 \cdot 10^{10}$ N/m^2. Foundation material is gravel with modulus $k_1 = 75$ MPa/m. Module of subgrade reaction $k = k_1 b = 1{,}875 \cdot 10^6$ Pa. Then parameter is $\alpha = 3{,}976$. Increasing (decreasing) of th is parameter can be linked with increasing (decreasing) of foundation stiffness k_1 (for constant flexural stiffness of a beam), whether decreasing (increasing) of flexural stiffness EI of a beam (for constant foundation stiffness).

3. Partially supported beam with two supporting segments.

3.1. Formulation and solution to the problem.

Here we formulate similar problem for the beam, foundation and load as it was above. The difference is that the internal part of the beam does not support on foundation (Figure 3.1). We consider the beam as a composite structure.

Table 2.1. Maximum deflections and bending moments.

Parameter	Deflections				Bending moments			
α \ v	1	0,8	0,6	0,4	1	0,8	0,6	0,4
1,257	0,1	0,197	0,961	4,79	0	0,039	0,102	0,19
2,236	0,01	0,031	0,11	0,512	0	0,038	0,101	0,19
3,976	0,001	0,0057	0,0236	0,084	0	0,032	0,095	0,19

This one consists of three homogenous segments supported on step foundation with partially constant stiffness: $k_1 = k_3 = k,\ k_2 = 0$.

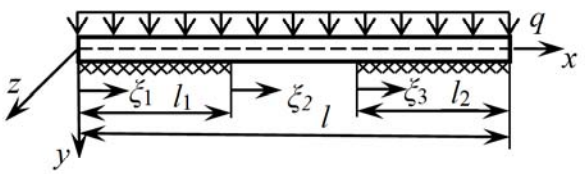

Figure 3.1. Calculation scheme of partially supported beam with two supporting segments.

Let us introduce dimensionless variables and parameters

$$\xi_i = \frac{x_i}{l};\ w_i = \frac{v_i}{l}(i = 1,2,3),\ v_1 = \frac{l_1}{l},\ v_2 = \frac{l_2}{l}$$

Deflection for the first segment $(0 \le \xi_1 \le v_1)$ and the third segment $(0 \le \xi_3 \le v_2)$ are described by solution (2.2) to equation (2.1), and state of cross sections ξ_1 and ξ_3 is described by matrix equation respectively:

$$\overline{w}_1(\xi_1) = V_1(\xi_1)\overline{W}_{10} + \overline{q}\,V_{1q}(\xi_1), \tag{3.1}$$

$$\overline{w}_3(\xi_3) = V_1(\xi_3)\overline{W}_{30} + \overline{q}\,V_{1q}(\xi_3), \tag{3.2}$$

and for the second segment $(0 \le \xi_2 \le 1 - v_1 - v_2)$ it is described by solution (2.4) to equation (2.3). State of a cross section ξ_2 of the second segment

$$\overline{w}_2(\xi_2) = V_2(\xi_2)\overline{W}_{20} + \overline{q}\,\overline{V}_{2q}(\xi_2), \tag{3.3}$$

Using conjunction conditions for segments: first and second

$$\overline{w}_1(v_1) = \overline{w}_2(0).$$

And second with third

$$\overline{w}_2(1 - v_1 - v_2) = \overline{w}_3(0).$$

We express initial parameters of the second and third segments through the initial parameters of the first segment

$$\overline{w}_{20} = V_1(v_1)\overline{W}_{10} + \overline{q}\,\overline{V}_{1q}(v_1), \tag{3.4}$$

and

$$\begin{aligned}\overline{w}_{30} &= V_2(1 - v_1 - v_2)V_1(v_1)\overline{W}_{10} + \\ &+ \overline{q}\left[V_2(1 - v_1 - v_2)\overline{V}_{1q}(v_1) + \overline{V}_{2q}(1 - v_1 - v_2)\right]\end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) to formulas (3.2) and (3.3) respectively, we obtain expression for state vector of all segments through the initial parameters of the first segment (two parameters are known: $w''_{10} = w'''_{10} = 0$).

$$\begin{aligned}\overline{w}_2(\xi_2) &= V_2(\xi_2)V_1(v_1)\overline{W}_{10} + \\ &+ \overline{q}\left[V_2(\xi_2)\overline{V}_{1q}(v_1) + \overline{V}_{2q}(\xi_2)\right]\end{aligned} \tag{3.6}$$

$$\begin{aligned}\overline{w}_3(\xi_3) &= V_1(\xi_3)V_2(1 - v_1 - v_2)V_1(v_1)\overline{W}_{10} + \\ &+ \overline{q}\left[V_1(\xi_3)V_2(1 - v_1 - v_2)\overline{V}_{1q}(v_1) + \right. \\ &\left. + V_1(\xi_3)\overline{V}_{2q}(1 - v_1 - v_2) + \overline{V}_{1q}(\xi_3)\right].\end{aligned} \tag{3.7}$$

Further, using matrix representation of state vectors for three segments (3.1), (3.6), (3.7), boundary conditions

$$w_1''(0) = w_1'''(0) = 0$$
$$w_3''(v_2) = w_3'''(v_2) = 0 \qquad (3.8)$$

And extended form of the equation (3.7) (similar to equation (2.9)), we obtain, satisfying to the second condition pair (3.8), system of equations for determining initial parameters w_{10} and w'_{10}:

$$\begin{cases} -4\alpha^4\left(K_2(\alpha v_2)(Z_1+Y_1+U_1)+K_1(\alpha v_2)(Z_2+Y_2+U_2)\right)+K_4(\alpha v_2)(Z_3+Y_3+U_3)+ \\ \qquad +K_3(\alpha v_2)(Z_4+Y_4+U_4)+K_2(\alpha v_2)=0 \\ -4\alpha^4\left(K_3(\alpha v_2)(Z_1+Y_1+U_1)+K_2(\alpha v_2)(Z_2+Y_2+U_2)\right)+K_1(\alpha v_2)(Z_3+Y_3+U_3)+ \\ \qquad +K_4(\alpha v_2)(Z_4+Y_4+U_4)+K_3(\alpha v_2)=0 \end{cases} \qquad (3.9)$$

Further, we determine deflection for segments

$$w_1(\xi_1) = K_4(\alpha\xi_1)w_0 + K_3(\alpha\xi_1)w'_0 + q\frac{1-K_4(\alpha\xi_1)}{4\alpha^4} \qquad (3.10)$$

$$\begin{aligned} w_2(\xi_2) = &\left(K_4(\alpha v_1) - 4\alpha^4\left(\xi_2 K_1(\alpha v_1) + \frac{\xi_2^2}{2}K_2(\alpha v_1) + \frac{\xi_2^3}{6}K_3(\alpha v_1)\right)\right)w_0 + \\ &+\left(K_3(\alpha v_1) + \xi_2 K_4(\alpha v_1) - 4\alpha^4\left(\frac{\xi_2^2}{2}K_1(\alpha v_1) + \frac{\xi_2^3}{6}K_2(\alpha v_1)\right)\right)w'_0 + \\ &+\overline{q}\left(\frac{1-K_4(\alpha v_1)}{4\alpha^4} + \xi_2 K_1(\alpha v_1) + \frac{\xi_2^2}{2}K_2(\alpha v_1) + \frac{\xi_2^3}{6}K_3(\alpha v_1) + \frac{\xi_2^4}{24}\right) \end{aligned} \qquad (3.11)$$

$$\begin{aligned} w_3(\xi_3) = &K_4(\alpha\xi_3)Z_1 + K_3(\alpha\xi_3)Z_2 + K_2(\alpha\xi_3)Z_3 + K_1(\alpha\xi_3)Z_4 + \\ &+\overline{q}\left(\begin{aligned} &K_4(\alpha\xi_3)Y_1 + K_3(\alpha\xi_3)Y_2 + K_2(\alpha\xi_3)Y_3 + K_1(\alpha\xi_3)Y_4 + \\ &+K_4(\alpha\xi_3)U_1 + K_3(\alpha\xi_3)U_2 + K_2(\alpha\xi_3)U_3 + K_1(\alpha\xi_3)U_4 + \frac{1-K_4(\alpha\xi_3)}{4\alpha^4} \end{aligned}\right); \end{aligned} \qquad (3.12)$$

where

$$\begin{aligned} Z_1 = &w_{10}\left(K_4(\alpha v_1) - 4\alpha^4\left(U_4K_1(\alpha v_1) + U_3K_2(\alpha v_1) + U_2K_3(\alpha v_1)\right)\right) + \\ &+w'_{10}\left(K_3(\alpha v_1) + U_4K_4(\alpha v_1) - 4\alpha^4\left(U_3K_1(\alpha v_1) + U_2K_2(\alpha v_1)\right)\right); \end{aligned}$$

$$\begin{aligned} Z_2 = &-4\alpha^4 w_{10}\left(K_1(\alpha v_1) + U_4K_2(\alpha v_1) + U_3K_3(\alpha v_1)\right) + \\ &+w'_{10}\left(K_4(\alpha v_1) - 4\alpha^4\left(U_4K_1(\alpha v_1) + U_3K_2(\alpha v_1)\right)\right); \end{aligned}$$

$$\begin{aligned} Z_3 = &-4\alpha^4\left(w_{10}\left(K_2(\alpha v_1) + U_4K_3(\alpha v_1)\right) + \right. \\ &\left.+w'_{10}\left(K_1(\alpha v_1) + U_4K_2(\alpha v_1)\right)\right); \end{aligned}$$

$$Z_4 = -4\alpha^4\left(w_{10}K_3(\alpha v_1) + w'_{10}K_2(\alpha v_1)\right);$$

$$Y_1 = \frac{1-K_4(\alpha v_1)}{4\alpha^4} + U_4K_1(\alpha v_1) + U_3K_2(\alpha v_1) + U_2K_3(\alpha v_1);$$

$$Y_2 = K_1(\alpha v_1) + U_4K_2(\alpha v_1) + U_3K_3(\alpha v_1);$$

$$Y_3 = K_2(\alpha v_1) + U_4K_3(\alpha v_1);$$

$$Y_4 = K_3(\alpha v_1);$$

$$U_1 = \frac{(1-v_1-v_2)^4}{24}; U_2 = \frac{(1-v_1-v_2)^3}{6};$$

$$U_3 = \frac{(1-v_1-v_2)^2}{2}; \; U_4 = 1-v_1-v_2.$$

3.2. Numerical results.

Using Maple, we provide calculations of dimension less deflections $w(\xi)$ and bending moments $w''(\xi)$ for beam supported on elastic foundation and loaded with evenly distributed static load $\overline{q}=1$ for different combinations of mechanical and geometric characteristic of a system "beam – foundation". At the same time, the middle part does not support on foundation in contrast with mentioned above case. For all cases end soft he beam are free and value of evenly distributed dimensionless load is $\overline{q}=1$. Figures 3.2 – 3.8 present calculation results as relative diagrams.

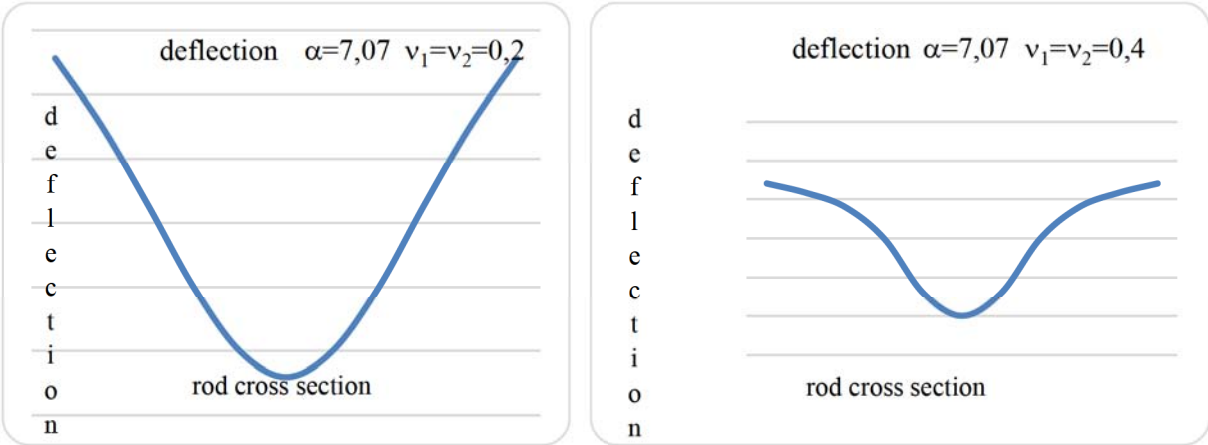

Figure 3.2. Beam deflection for system stiffness value $\alpha = 7{,}07$.

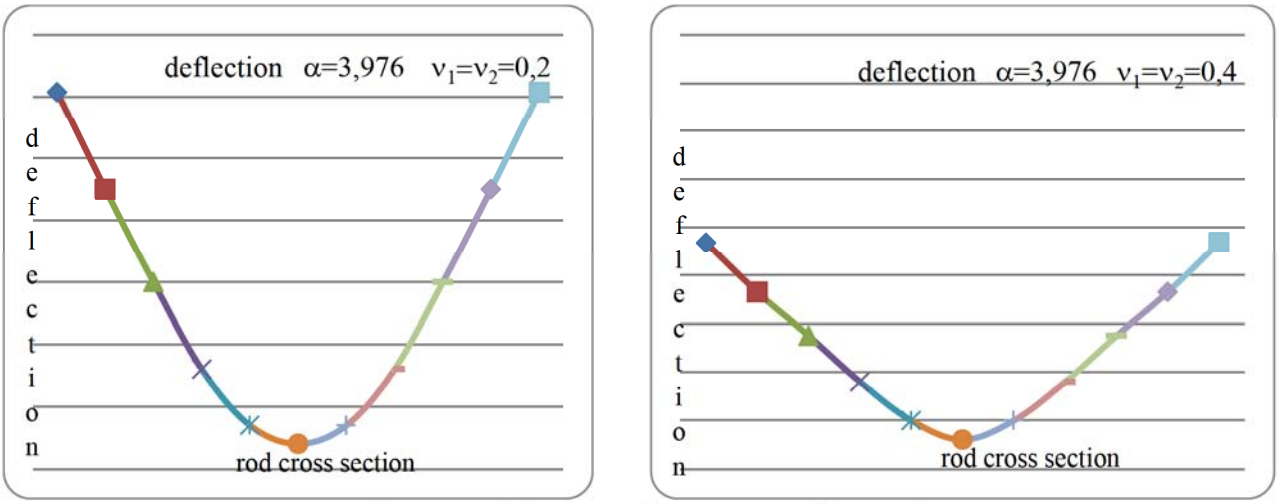

Figure 3.3. Deflections of the beam for system stiffness value $\alpha = 3{,}976$.

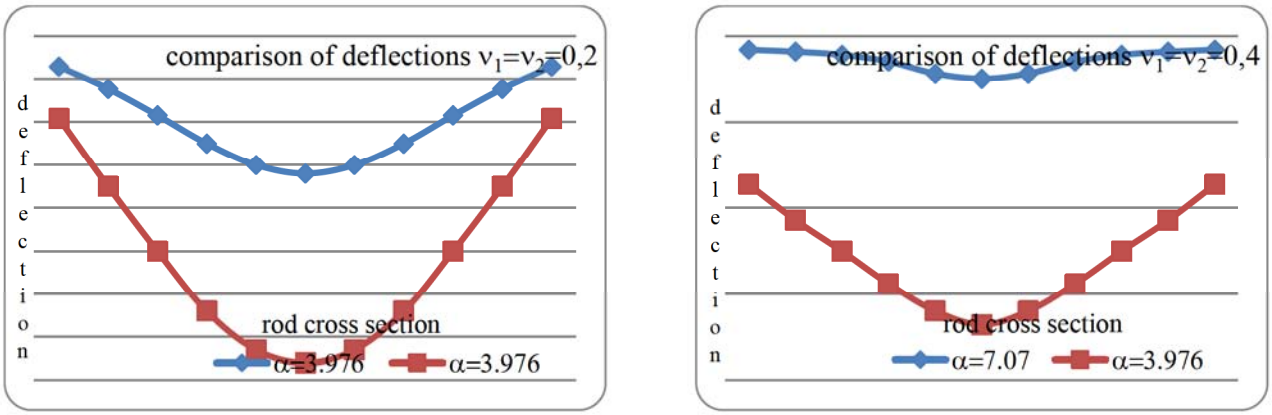

Figure 3.4. Comparison of deflections by attribute of damage length Δ.

Figures 3.2-3.4 show diagrams of beam deflections for symmetric location of supported section of equal length $(\nu_1 = \nu_2)$ and different values of free segment length simulating damage $\Delta = 1 - \nu_1 - \nu_2$ for two values of generalized stiffness of a system "beam – foundation": $\alpha = 7{,}07;\ 3{,}976$. Figure 3.2 shows diagrams of deflection for system stiffness value $\alpha = 7{,}07$.

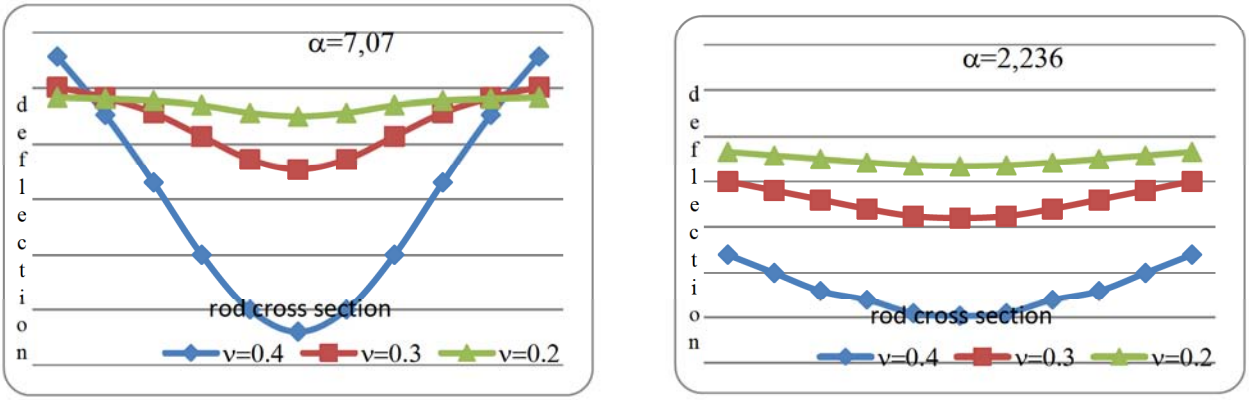

Figure 3.5. Comparison of deflections by attribute of generalized stiffness.

Let us note that relatively to parameter α of generalized stiffness of a system "beam – foundation" its value $\alpha = 7{,}07$ corresponds, for example, to reinforced concrete beam with length of $l = 6{,}7$ m, rectangular cross section with sides: width $b = 0.25$ m, height $h = 0.18$ m, inertia moment of a cross section is $I = 1{,}215 \cdot 10^{-4} m^4$, Yung modulus is $E = 3.05 \cdot 10^{10} N/m^2$. The beam supports on ballast layer of crushed stones with modulus $k_1 = 7.5$ MPa/m. Module of subgrade reaction is $k = k_1 b = 1{,}875 \cdot 10^6$ Pa. At the same time

$$\alpha = \sqrt[4]{\frac{kl^4}{4EI}} = 7{,}07 \ .$$

Increasing (decreasing) of parameter α value for given length of the beam may be reached, for example, at increasing (decreasing) of foundation stiffness k (for constant beam stiffness EI), whether with decreasing (increasing) of flexural stiffness of a beam EI (for constant foundation stiffness k). Figure 3.3 presents deflections of the beam for system stiffness value $\alpha = 3{,}976$.

In order to compare, Figure 3.4 presents diagrams of deflections in relation with damage length Δ for different values of system stiffness α.

In order to compare, Figure 3.5 presents diagrams of deflection for attribute of generalized system stiffness for different values of damage length Δ.

Figure 3.6 shows diagram of beam deflections for system with high generalized stiffness $\alpha = 9{,}236$ for damages section length Δ=0,4.

Figure 3.7 presents comparison of diagrams of bending moments by attribute of damaged section length for generalized stiffness $\alpha = 2{,}236$.

Figure 3.8 shows diagrams of deflections and bending moments for the case of non-symmetric sections' location and non-equal length of it.

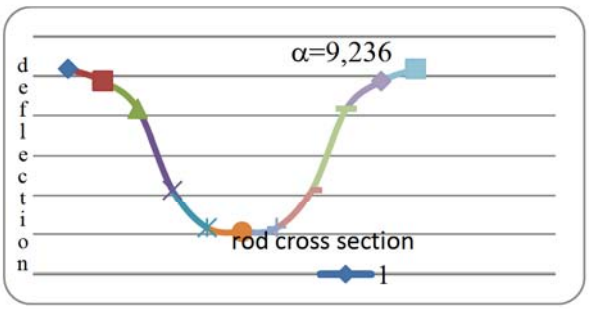

Figure 3.6. Diagram of beam deflection for high generalized stiffness $\alpha = 9{,}236$.

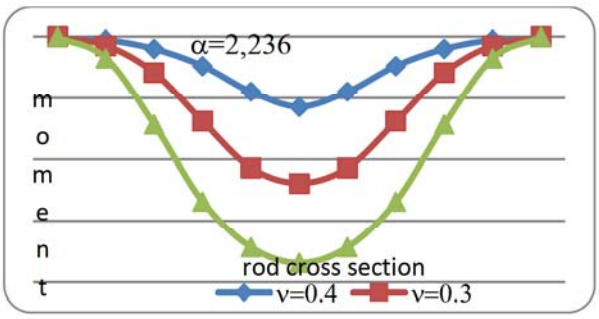

Figure 3.7. Diagrams of bending moments for $\alpha = 2{,}236$.

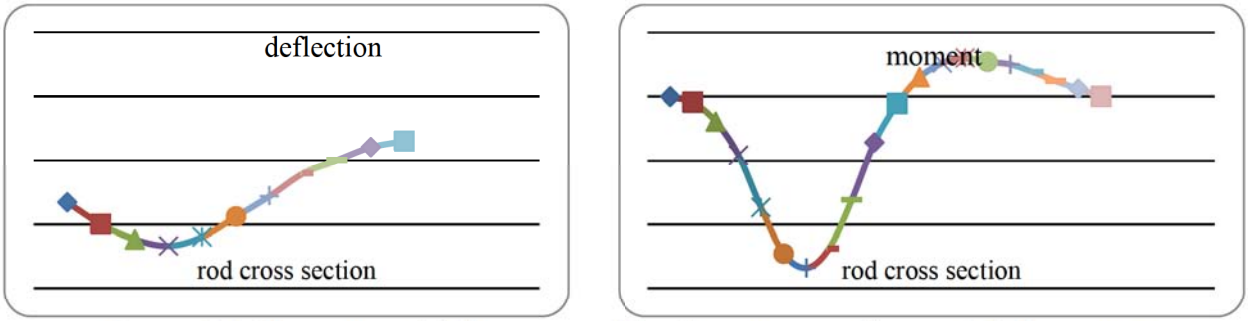

Figure 3.8. Diagrams of deflections and bending moments for $\alpha = 3{,}976$, $v_1 = 0{,}2$ *and* $v_2 = 0{,}6\ (\Delta = 0{,}2)$.

Numerical analysis shows that character and values of deflections and bending moments for symmetrically supported beams significantly depend on damage value $\Delta = 1 - v_1 - v_2$ and generalized stiffness of system "beam foundation". For all values of stiffness from considered range $2{,}236 \le \alpha \le 7{,}07$ and damage sizes $0{,}2 \le \Delta \le 0{,}6$, deflection functions are convex (diagrams of deflections are plotted in tension zones). For high values of generalized stiffness $\alpha > 7{,}07$ and large damages $\Delta > 0{,}6$, inflections appear and deflection changes it direction. Values of maximum deflections increase for all stiffness values (Figures 3.3, 3.4). For supported segments of different length (Figure 3.8), moments change it sign (diagram in Figure 3.8,b is plotted in tension zones).

CONCLUSIONS

Suggested mathematic deformation model for beam partially supported on elastic foundation is constructed using the initial parameters' method and allows to obtain dimensionless analytical solution to quasi-static universal problems for different variant of boundary conditions at sudden appearing of foundation damage (partial foundation failure).

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Vladimir I. Travush, Full Member of the Russian Academy of Architecture and Construction Sciences, Professor, Dr.Sc., Vice-President of the Russian Academy of Architecture and Construction Sciences; 107031, Moscow, st. Bolshaya Dmitrovka, 24/1;
phone/fax +7 (495) 625-79-67; e-mail: travush@mail.ru.

Vladimir A. Gordon, Advisor of the Russian Academy of Architecture and Construction Sciences, Professor, Dr.Sc., Head of the Department of Higher Mathematics; Orel State University named after I.S. Turgenev; 302026, Orel, st. Komsomolskaya, 95; phone/fax +74862419802;
E-mail: gordon@ostu.ru.

Vitaly I. Kolchunov, Full Member of the Russian Academy of Architecture and Construction Sciences, Professor, Dr.Sc., Head of Department of Unique Buildings and Structures; Southwest State University (SWSU);; 305040, Russia, Kursk, st. 50 let Oktyabrya, 94;
phone/fax +74712222461; e-mail: asiorel@mail.ru.

Evgeny V. Leontiev, Head of the Department of Constructive Reliability and Safety of Main State Expertise of Russia; 119049, Moscow, st.Bolshaya Yakimanka, 42/1-2; phone/fax +74956259595;
E-mail: e.leontyev@gge.ru.

Травуш Владимир Ильич, академик РААСН, профессор, доктор технических наук, вице-президент Российской академии архитектуры и строительных наук; 107031, Москва, ул. Большая Дмитровка, д.24, стр.1; тел./факс +7(495)625-79-67; e-mail: travush@mail.ru.

Гордон Владимир Александрович, советник РААСН, профессор, доктор технических наук, профессор кафедры технической физики и математики Орловского государственного университета имени И.С. Тургенева; 302026, г. Орел, ул. Комсомольская, 95;
тел./факс +74862419802; e-mail: gordon@ostu.ru.

Колчунов Виталий Иванович, академик РААСН, профессор, доктор технических наук, заведующий кафедрой уникальных зданий и сооружений Юго-Западного государственного университета; 305040, Россия, г. Курск, ул. 50 лет Октября, 94;
тел./факс +74712222461; e-mail: asiorel@mail.ru.

Леонтьев Евгений Владимирович, начальник отдела конструктивной надежности и безопасности объектов Главгосэкспертизы России; 119049, г. Москва, ул. Большая Якиманка, 42, стр. 1-2;
тел./факс +74956259595; e-mail: e.leontyev@gge.ru.