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SET OF EQUATIONS OF AFEM AND PROPERTIES OF ADDITIONAL FINITE ELEMENTS

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Abstract: The paper considers the action of additional finite elements on the main set of linear equations when developed Additional Finite Element Method (AFEM) is used for analysis of structures with several nonlinear properties at limit states and failure models. AFEM is a variant of Finite Element Method (FEM), which adds to traditional sequence of solution of problem by FEM the units of two well-known methods of structural analysis: method of additional loads and method of ultimate equilibrium. AFEM suggests the additional finite elements and additional design diagrams for gradually transformation of the main set of equilibrium equations at the first step of loading to this set at the last one according to ideal failure model of structure.

Keywords: additional finite element method, limit state, ideal failure model, additional finite element, additional design diagram, equilibrium equation

СИСТЕМА УРАВНЕНИЙ МЕТОДА ДОПОЛНИТЕЛЬНЫХ КОНЕЧНЫХ ЭЛЕМЕНТОВ И СВОЙСТВА ДОПОЛНИТЕЛЬНЫХ КОНЕЧНЫХ ЭЛЕМЕНТОВ

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Аннотация: Статья рассматривает воздействие дополнительных конечных элементов на основную систему линейных уравнений, когда разрабатываемый метод дополнительных конечных элементов (МДКЭ) используется для расчета конструкций с несколькими нелинейными свойствами по предельным состояниям и моделям разрушения. МДКЭ является вариантом метода конечных элементов (МКЭ), который добавляет к традиционной последовательности решения задачи операции двух широко известных методов расчета конструкций: метода дополнительных нагрузок и метода предельного равновесия. МДКЭ предлагает дополнительные конечные элементы и дополнительные расчетные схемы для постепенного преобразования основной системы уравнений равновесия на первом шаге нагружения в систему на последнем шаге в соответствии с идеальной моделью разрушения конструкции.

Ключевые слова: метод дополнительных конечных элементов, предельное состояние, идеальная модель разрушения, дополнительный конечный элемент, дополнительная расчетная схема, уравнение равновесия

INTRODUCTION

Realization of nonlinear analysis according to method of ultimate equilibrium [1] is impossible without taking account of all physical nonlinear properties exhibited by the structure up to the moment when the ultimate limit state (state of ultimate equilibrium) is reached. The developed Additional Finite Element Method (AFEM) [2, 3, 4] is destined for solving of this problem.

1. GENERAL INFORMATION OF AFEM

Additional finite element method (AFEM) is a variant of finite element method (FEM) destined for nonlinear analysis of plane and space structures at limit state.

This method adds to traditional sequence of solving problem by means of FEM the elements of the method of ultimate equilibrium (limit states) and the method of elastic solutions (method of additional loads).

2. SET OF EQUATIONS OF FEM AND MAIN PROBLEM

The main operation of FEM [5] is the solving of the set of algebraic equation:

$$KV = P \tag{1}$$

where V = matrix of unknown node displacements; P = matrix of external load; K = stiffness matrix of considered structure is formed from coefficients of stiffness matrices of the separate finite elements (FE).

The set of equations (1) solves one time in linear analysis because of matrix $K = const.$

In nonlinear analysis the set of equation (1) looks like this:

$$K_{nonl}V = P \tag{2}$$

where K_{nonl} = stiffness matrix of structure with nonlinear properties which is changed in accordance with the degree of influence of these properties.

In this case the set of equations (2) must be solved by iterative process. In this process matrix K ($K \neq const.$) turns into matrix K_{nonl} gradually.

The transformation of the set of equation (1) into set of equation (2) is connected with difficulties in presence of several of physical nonlinear properties due to its different causes and effects.

Usually the structure has n types of nonlinear properties at the end of its operating period before collapse.

Due to different defects the matrix K_{nonl} gradually decreases from K to K_{min}, where K_{min} = its minimal value:

$$K>K_1>K_2>K>\ldots>K_i>\ldots>K_{n-1}>K_n=K_{min} \tag{3}$$

where K_i = stiffness matrix of structure with i nonlinear properties (i changes from 1 to n).

Each nonlinear property appears under corresponding value of increasing load P. Before collapse the external load P reaches its maximal value P_{max}:

$$P_0<P_1<P_2<P_3<\ldots< P_i<\ldots<P_{n-1}< P_n<P_{max} \tag{4}$$

where P_1, P_2, P_3, P_i, P_{i+1} = values of external load P, when the first, the second, the third, the i-th and the (i+1)-th nonlinear properties appear respectively. Thus the structure reveals i-th nonlinear property under external load $P_i < P < P_{i+1}$.

Therefore the set of equations (2) ought to be created under the conditions (3) and (4) and the must change its form depend on number of nonlinear properties at considered stage and value of load P.

So, the set of equations (2) has view (1) in linear behavior under $P_0 < P < P_1$ and $i = 0$. Then in the beginning and continuing of nonlinear behavior the set of equations changes gradually:

under $P_1 < P < P_2$ and one nonlinear property ($i = 1$):

$$K_1V = P \tag{5}$$

under $P_2 < P < P_3$ and two nonlinear properties ($i = 2$):

$$K_2V = P \tag{6}$$

under $P_3 < P < P_4$ and three nonlinear properties ($i = 3$):

$$K_3V = P \tag{7}$$

and so on

under $P_i < P < P_{i+1}$ and i nonlinear properties:

$$K_i\,V = P \tag{8}$$

and so on
under $P_{n-1} < P < P_n$ and (n-1) nonlinear properties ($i = n$-1):

$$K_{n-1}V = P \tag{9}$$

under $P_n < P < P_{max}$ and n nonlinear properties ($i = n$):

$$K_n V = P \tag{10}$$

The set of equation must be formed in order to turn matrix K into matrix $K_n = K_{min}$ and external load P into P_{max} due to n nonlinear properties of structures.
It must be realized according to formulas (5)–(10) under conditions (3) and (4).
Such realization is the difficult problem.

3. METHOD ADDITIONAL LOADS (METHOD OF ELASTIC DECISIONS)

This method was suggested for solving of deformation problems of plastic theory by A.A. Ilyushin [6].
This method uses for analysis of structures at plastic behavior when the main of equations has form (2).
Matrix K_{nonl} is stiffness matrix of structure with plastic property which is changed in accordance with the level of influence of this property. The next step is based on separation from this matrix of its linear and nonlinear parts [7]:

$$K_{nonl} = K + \Delta K_{nonl} \tag{11}$$

where K and ΔK_{nonl} = the linear and nonlinear components of matrix K_{nonl} respectively.
If we substitute the formula (11) in the expression (2) we may get the next equation:

$$(K_{nonl} + \Delta K_{nonl})V = P \tag{12}$$

The next step is removing the parentheses and the transferring the second part in right:

$$KV = P - \Delta K_{nonl}V \tag{13}$$

where $(-\Delta K_{nonl}V)$ = the value of additional load in carrying out of the iterative process.
This method allows the solving the set of linear equations (2) with constant coefficients in the left hand side and the obtaining of inverse matrix K^{-1} only once.
The simplicity is main advantage of Method of Additional loads (Method of Elastic Decisions).
It is most prevalent and efficient method of nonlinear analysis of structures with plastic properties.

4. IDEAL FAILURE MODEL AND THE SET EQUATIONS OF AFEM

The criterion of collapse of the structure must be determined according to its nonlinear properties. It is necessary to know the limit of operating period of considered structure.
The Theory Ultimate Equilibrium and Ultimate Equilibrium Method (Ultimate Limit State Method) may be used for introduction of such criterion.
According to this method the structure reaches the state of ultimate equilibrium or ultimate limit state before collapse.
In this stage the external load P_{lim} is maximal. Therefore the condition (4) is fulfilled:

$$P_{lim} = P_{max} \tag{14}$$

At ultimate limit state the stiffness matrix of the structure K_{lim} is minimal because all nonlinear properties are manifested.
Thus the condition (3) is fulfilled too:

$$K_{lim} = K_n = K_{min} \tag{15}$$

The definition of this stiffness matrix K_{lim} and its intervening values $K_1, K_2, K_3, \dots, K_i, \dots K_{n-1}, K_n$ is difficult problem.
For it decision the ideal failure model is suggested by author [2, 3, 4].
This model is the design diagram of the structure at the ultimate limit state or state of

ultimate equilibrium, i.e. moment previous of collapse.

At one hand the ideal failure model of structure must correspond to possible real failure model and at the other hand to main parameters of initial design diagram.

It helps to solve some problems for analysis of structures with several nonlinear properties [8, 9].

The point is that the initial design diagram must change step-by-step loading in accordance with nonlinear properties appeared as the ultimate limit state is reached.

As a result an initial design diagram transforms into an ideal failure model of the considered structure.

At ultimate limit state the set of equations (1) and (2) ought to have form:

$$K_{lim}\,V = P_{lim} \qquad (16)$$

Thus stiffness matrix K_{lim} of structures with all nonlinear properties at ultimate limit state must form on the basis of ideal failure model. Iterative process must go under condition (3).

Additional Finite Element Method determines that at the moment of limit state the next equation is correct:

$$K_{lim}=K+\Delta K_1+\Delta K_2+\ldots+\Delta K_i+\ldots+\Delta K_n \qquad (17)$$

Here each component ΔK_i is stiffness matrix of the i-th additional design diagram consisting of additional finite elements taking into account the i-th nonlinear property (i changes from 1 to n) and may be solved according to next formula:

$$\Delta K_i = K_i - K_{i-1} \qquad (18)$$

where K_i = stiffness matrix of structure with regard for the i-th nonlinear property; K_{i-1} = stiffness matrix of structure without regard for the i-th nonlinear property.

Introduction of ideal failure model allows realization of step-by-step analysis with gradual taking into account of influence of several (n) nonlinear properties due to determination of criterion of limit state of structure before collapse.

Some examples of ideal failure models of structures are given in [10, 11].

For gradual transformation of the set of equation according to (5) AFEM suggests to develop some operations of Method of Additional loads (Method of Elastic Decision) and use additional design diagrams consisting of additional finite elements (AFE-s) [12, 13].

In this case the sets of equations (5) – (10) are formed according to the formula (13) under conditions (3) and (4):

under $P_1 < P < P_2$ and one nonlinear property ($i = 1$):

$$KV = P - \Delta K_1 V \qquad (19)$$

under $P_2 < P < P_3$ and two nonlinear properties ($i = 2$):

$$KV = P - \Delta K_1 V - \Delta K_2 V \qquad (20)$$

under $P_3 < P < P_4$ and three nonlinear properties ($i = 3$):

$$KV = P - \Delta K_1 V - \Delta K_2\,V - \Delta K_3 V \qquad (21)$$

and so on

under $P_i < P < P_{i+1}$ and i nonlinear properties:

$$KV=P-\Delta K_1V-\Delta K_2V-\Delta K_3V-\ldots-\Delta K_iV \qquad (22)$$

and so on

under $P_{n-1} < P < P_n$ and (n-1) nonlinear properties ($i = n$-1):

$$KV=P-\Delta KV-\Delta K_2V-\Delta K_3V-\ldots-\Delta K_iV-\ldots -\Delta K_{n-1}V \qquad (23)$$

under $P_n < P < P_{max}$ and n nonlinear properties ($i = n$):

$$KV=P-\Delta K_1V-\Delta K_2V-\Delta K_3V-\ldots-\Delta K_iV-\ldots -\Delta K_{n-1}V-\Delta K_nV \qquad (24)$$

Here ΔK_1 = stiffness matrix of the first additional design diagram consisting of additional finite elements taking into account the first nonlinear property; ΔK_2 = stiffness matrix of the second additional design diagram consisting of additional finite elements taking into account the second nonlinear property; ΔK_3 = stiffness matrix of the third additional design diagram consisting of additional finite elements taking into account the third nonlinear property; ΔK_i = stiffness matrix of the i-th additional design diagram consisting of additional finite elements taking into account the i-th nonlinear property; ΔK_{n-1} = stiffness matrix of the (n–1)-th additional design diagram consisting of additional finite elements taking into account the (n–1)-th nonlinear property; ΔK_n = stiffness matrix of the n-th additional design diagram consisting of additional finite elements taking into account the n-th nonlinear property.

Each additional design diagram is a geometrical replica of the initial design diagram but it is destined for gradual transformation of an initial design diagram into design diagram with all n nonlinear properties.

Additional design diagram may be compared with empty space imbedded in the initial design diagram and filled negative stiffness for nonlinear analysis at ultimate limit states.

In relations (19) – (24) every value ($-\Delta K_i V$) determines the influence of i-th nonlinear property.

For example the term ($-\Delta K_1 V$) of the left-hand part of these equations is the additional load which with the main load P must be applied to linear structure to reach the displacements corresponding to its displacements with the first nonlinear property under the action of the only external load P.

Thus in nonlinear analysis the sets of algebraic equations (5) – (10) take the forms (19) – (24) and provide taking into account the influence of each nonlinear property.

This way is corresponded to logic of FEM and allows to using of different theoretical data [14,15,16] for nonlinear computer analysis [15, 17] according to normative requirements [18].

5. ADDITIONAL FINITE ELEMENTS AND THREE WAYS FOR CALCULATION OF ADDITIONAL LOADS

Every additional design diagram takes into account only one nonlinear property and consists of corresponding additional finite elements (AFE-s) [19].

Additional finite elements (AFE-s) are recommended for gradual transformation of the initial finite elements with linear properties into the same finite elements but with nonlinear properties which correspond to the reached stage of their limit states.

The scheme of its action of additional finite element (AFE) is

FE with nonlinear property =
= FE without nonlinear property +
+ AFE for taking into account
the nonlinear property.

It is necessary to know mathematical relationships which characterize the properties of additional finite elements for their application in nonlinear analysis of structures at limit state.

These relationships are necessary when the operations of tree numerical methods (elastic decisions (additional loads), additional stresses and additional strains) are used. Such approach is determined by requirements of realization of nonlinear design of the structure at limit state as such design lead to design of physical n-nonlinear systems.

Since the properties of additional finite elements are determined by the properties of a corresponding main finite element then the desired mathematical relationships are determined by analogous relationships of the main finite element:

1) Relationship between node reactions and displacements $\Delta R_e = f(V)$;
2) Formula for determination of stiffness matrices ΔK_e;
3) Formula for determination of node reactions ΔR_e;

4) Relationship between node reactions and stresses $\Delta R_e = f(\sigma)$;
5) Relationship between node reactions and strains $\Delta R_e = f(\Delta\varepsilon)$;
6) Formula for determination of stresses $\Delta\sigma$;
7) Formula for determination of strains $\Delta\varepsilon$;
8) Formula for determination of additional load F_e.

Last formula is necessary when additional load substitutes the action of additional finite element according to next scheme:

FE with nonlinear property =
= FE without nonlinear property +
+ result of action of additional load F_e
for taking into account the nonlinear property.

The appointment of additional finite element is the change of stress-strain state of the main finite element without allowance for nonlinear property to the level of stress-strain state which is appeared in the same finite element with allowance for this property.

Such approach opens the opportunity to use of the two ways of change of properties of the main finite element in view of appearance of the definite nonlinear property: change of its initial stress state and change of its initial strain state.

Since the additional finite element changes the properties of the main finite element then it is recommended to use two types of additional finite elements for realization of each of the two indicated ways:

1) additional finite element of the first type changes a stress state of the main finite element and it does not change its strain state;
2) additional finite element of the second type changes a strain state of the main finite element and it does not change its stress state.

It is known that stiffness matrix of any finite element connects its node forces and displacements.

Thus for the main finite element at definite stage of its behavior at particular limit state for the allowance of the i-th nonlinear property this relation looks like this:

$$K_{nonl,e,i}\, V = R_{nonl,e,i} \qquad (25)$$

where $R_{nonl,e,i}$ = node reactions in the finite element for the allowance of the i-th nonlinear property; V = node displacements; $K_{nonl,e,i}$ = stiffness matrix of finite element for the allowance of the i-th nonlinear property.

In order to further use of operations of the method of elastic decisions let us express the stiffness matrix $K_{nonl,e,i}$ of the finite element with nonlinear properties in the next form:

$$K_{nonl,e,i} = K_{nonl,e,i-1} + \Delta K_{nonl,e,i} \qquad (26)$$

where $K_{nonl,e,i-1}$ = stiffness matrix of the main FE without allowance of the i-th nonlinear property;

$\Delta K_{nonl,e,i}$ = stiffness matrix of additional finite element destined for the allowance of the degree of influence of the i-th nonlinear property on behavior of this element. So that it is determined by means of next formula:

$$\Delta K_{nonl,e,i} = K_{nonl,e,i} - K_{nonl,e,i-1} \qquad (27)$$

In general the stiffness matrix of AFE $\Delta K_{nonl,e,i}$ is not equal to 0, i.e.

$$\Delta K_{nonl,e,i} \neq 0 \qquad (28)$$

If the influence of any i-th nonlinear property on behavior of the main finite element is absent then the stiffness matrix of the corresponding AFE is:

$$\Delta K_{nonle,i} = 0 \qquad (29)$$

In taking account of the first nonlinear property the stiffness matrix of additional FE $\Delta K_{nonl,e,1}$ may be determined by the next formula:

$$\Delta K_{nonl,e,1} = K_{nonl,e,1} - K_{nonl,e,0} = K_{nonl,e,1} - K_e \qquad (30)$$

where K_e = stiffness matrix of the main FE with linear properties.

Node reactions $R_{nonl,i}$ of finite element for the allowance of the *i*-th nonlinear property may be expressed by reactions $R_{nonl,i-1}$ of the finite element without this *i*-th nonlinear property:

$$R_{nonl,i} = R_{nonl,i-1} + \Delta R_{nonl,i} \tag{31}$$

where $\Delta R_{nonl,i}$ = change of node reactions of finite element due to manifestation of the i-th nonlinear property.

The first component of the right-hand part of this expression presents node reactions in the finite element without allowance for the *i*-th nonlinear property, i.e.

$$K_{nonl,e,i-1}\, V = R_{nonl,i-1} \tag{32}$$

If we substitute (26) and (31) in (25) with allowance for (32) we obtain:

$$\Delta K_{nonl,e,i}\, V = \Delta R_{nonl,i} \tag{33}$$

This formula determines the dependence between node reactions and node displacements in additional finite element for the allowance of the *i*-th nonlinear property.

Its stiffness matrix $\Delta K_{nonl,e,i}$ is determined by formula (27) and for the allowance of the first nonlinear property it is determined by formula (30).

If the influence of any nonlinear property on behavior of the main finite element is absent then the stiffness matrix of the corresponding additional finite element is equal to 0 (29).

The approach to the way of formation of additional load F_e for the allowance of the nonlinear properties for both types of AFE is the same.

Scheme of method of elastic solutions (additional loads) is given at Figure 1. This scheme is used for the allowance of the first nonlinear property (plasticity).

Nevertheless there are special singularities for each type of additional finite element which simplify algorithm of problem solution in comparison with general case.

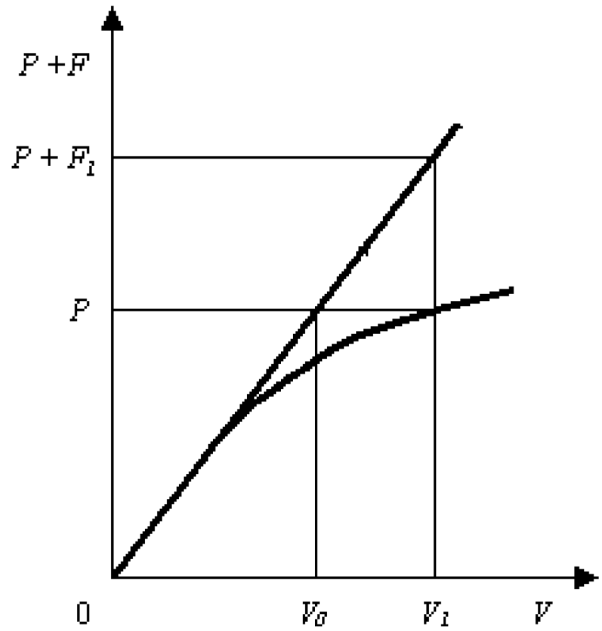

Figure 1. Scheme of the method of elastic decisions which is used by AFEM for the allowance of the first nonlinear property when its additional load is $F_{e,1} = -\Delta R_{nonl,1}$ [20,21].

If we compare formulae (31) and (33), we may make a conclusion that by means of node reactions of additional finite elements it is possible to form the vector of additional load for the main finite element, by means of which it is possible to allow for the considered nonlinear property.

This vector $F_{e,i}$ is determined by means of the next relationship:

$$F_{e,i} = -\Delta K_{nonl,e,i}\, V = -\Delta R_{nonl,i} \tag{34}$$

Further it follows that we come to a little stop on the case of formation of additional load for collapse of the main finite element after reaching of its ultimate limit state.

In this case let us note the stiffness matrix of additional finite element as ΔK_{lim} and then:

$$\Delta K_{lim} = -\Delta K \tag{35}$$

where K = stiffness matrix of the main finite element with linear properties.

Node reactions of additional finite element ΔR_{lim} are determined by next formula:

$$\Delta R_{lim} = -\,\Delta K_e\, V = -\,R \qquad (36)$$

where R = node reactions of finite element with linear properties. In this connection the additional load for collapse of finite element is equal to:

$$F_e = K_e\, V = R \qquad (37)$$

Actually it means that after collapse the main finite element ceases its existence and does not influence the neighboring elements and additional load guarantees this fact.

On the basis of these vectors $F_{e,i}$ the vector of additional load for the allowance of the i-th nonlinear property of the total structure according its design diagram may be formed. Then on the basis of separate vectors or each considered in analysis nonlinear property the total vector for all considered nonlinear properties may be formed.

The additional finite element of the first type does not change strain state. It changes only the stress state of the main finite element without account of particular nonlinear property up to the level of stress state of the same element with account of the given nonlinear property.

The absence of strains and the presence of stresses are characteristic for the first type of additional finite element, i.e

$$\Delta\varepsilon_{nonl,i} = 0 \qquad (38)$$

$$\Delta\sigma_{nonl,i} \neq 0 \qquad (39)$$

It means that if the additional finite element of the first type is used for the allowance of the i-th nonlinear property of the main finite element next formulae are correct for determination of stresses and strains:

$$\sigma_{nonl,i} = \sigma_{nonl,i1} + \Delta\sigma_{nonl,i} \qquad (40)$$

$$\varepsilon_{nonl,i} = \varepsilon_{nonl,i-1} \qquad (41)$$

These relationships for the allowance of the first nonlinear property look like these ones:

$$\sigma_{nonl,1} = \sigma + \Delta\sigma_{nonl,1} \qquad (42)$$

$$\varepsilon_{nonl,1} = \varepsilon \qquad (43)$$

It should be noted that general formula (34) for calculation of additional load $F_{e,i}$ for the main finite element is correct for both types of additional finite elements.

However besides this general approach there are individual cases of determination of vector of node reactions $\Delta R_{nonl,i}$ and consequently of additional load $F_{e,i}$ for both types of additional finite elements.

These individual cases are based on the relationships between node reactions with stresses and strains.

For additional finite element of the first type this individual case is based on relationship between node reactions and stresses.

So for the main finite element with linear properties the relationship between node reactions R and stresses σ looks like:

$$R = C\,\sigma \qquad (44)$$

Where C = matrix connecting elastic node reactions and stresses.

The analogous formula for the main finite element with linear properties looks like:

$$R_{nonl} = C\,\sigma_{nonl} \qquad (45)$$

Using the relationships (43) and (44) let us write down the formulae for determinations of node reactions $\Delta R_{nonl,i}$ in additional finite element of the first type for the allowance of the first nonlinear property:

$$\Delta R_{nonl,1} = C\,\Delta\sigma_{nonl,1} \qquad (46)$$

and for the allowance of the i-th nonlinear property:

$$\Delta R_{nonl,i} = C\,\Delta\sigma_{nonl,i} \qquad (47)$$

Substituting the expression (46) in (34) we obtain the formula for determination of the vector of additional load in general case:

$$F_{e,i} = -\,C\,\Delta\sigma_{nonl,i} \qquad (48)$$

In elimination of the main finite element after reaching of its ultimate limit state the formula looks like:

$$F_{e,i} = C\,\sigma \qquad (49)$$

where σ = stresses of finite element with linear properties.

This simplified variant of formation of the additional load using stresses $\Delta\sigma_{nonl,i}$ for the allowance of the plastic properties of concrete was developed in research.

It allows to use some operations of the method of additional stresses. Scheme of this process for the allowance of the first nonlinear property by means of additional finite element is shown at Figure 2.

Additional finite element of the second type does not change stress state.

It changes only stain state of the main finite element without allowance of the nonlinear property up to the level of strain state for the allowance of the given nonlinear property.

The dependence (33) connecting node reactions and node displacements as it reflects the main property of any additional finite element is correct for additional finite element of the second type too.

Stiffness matrix of this element may be determined by formula (27) and its additional load for the allowance of the nonlinear property may be determined by formula (34).

These expressions are correct for additional finite element of any type too.

Relating to other formulae describing the properties of additional finite element of the second type that it is necessary to use another approach for their obtaining based on the peculiarities of this element.

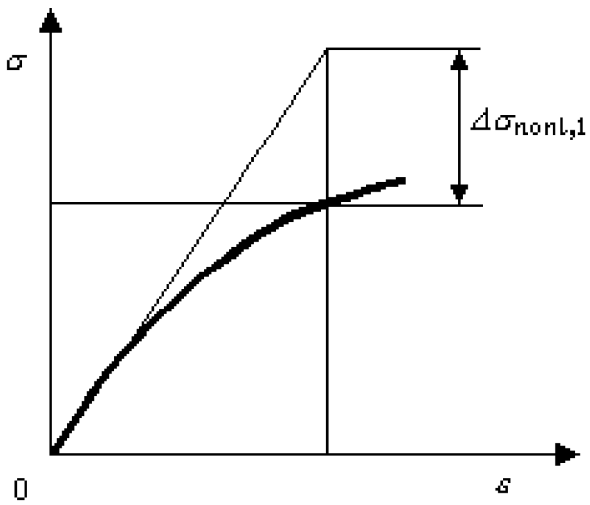

Figure 2. Scheme of the method of additional stresses which is used by AFEM for the allowance of the first nonlinear property by means of AFE of the first type [19, 20]

The absence of stresses and presence of strains is a defining feature of the additional finite element of the second type, i.

$$\Delta\varepsilon_{nonl,i} \neq 0 \qquad (50)$$

$$\Delta\sigma_{nonl,\,i} = 0 \qquad (51)$$

It means that if the additional finite element of the second type is used for the allowance of the *i*-th nonlinear property of the main finite element the next formulae for determination of stresses and strains are correct:

$$\sigma_{nonl,i} = \sigma_{nonl,i}\;\varepsilon_{nonl,i-1} \qquad (52)$$

$$\varepsilon_{nonl,i} = \varepsilon_{nonl,i-1} + \Delta\varepsilon_{nonl,i} \qquad (53)$$

They are correct for the first nonlinear property:

$$\sigma_{nonl,1} = \sigma(\varepsilon) \qquad (54)$$

$$\varepsilon_{nonl,1} = \varepsilon + \Delta\varepsilon_{nonl,1} \qquad (55)$$

Special case of determination of additional load for additional finite element of the second type is based on the relationship of node reactions and strains of the main finite element.

For the main finite element with linear properties this relationship has next form:

$$R = G\,\varepsilon \qquad (56)$$

where G = matrix connecting elastic node reactions and strains.
For the main finite element with nonlinear properties this relationship looks like:

$$R_{nonl} = G\,\varepsilon_{nonl} \qquad (57)$$

Taking into consideration formulae (56) and (57) the formula for determination of node reaction in additional finite element of the second type for the allowance of the first nonlinear property will be:

$$\Delta R_{nonl,1} = G\,\Delta\,\varepsilon_{nonl,1} \qquad (58)$$

This formula for the allowance of the i-th nonlinear property looks like:

$$\Delta R_{nonl,i} = G\,\Delta\varepsilon_{nonl,i} \qquad (59)$$

Using the formulae (59) and (34) we may obtain the formula for determination of the additional load at any stage:

$$F_{e,i} = -G\,\Delta\varepsilon_{nonl,i} \qquad (60)$$

In case of collapse of the main finite element when its ultimate limit state is reached this formula will be of next form:

$$F_e = G\varepsilon \qquad (61)$$

where ε = strains of the finite element with linear properties.
This variant of formation of additional load $F_{e,i}$ using the strains $\Delta\varepsilon_{nonl,i}$ of additional finite element is suitable to take into account residual strains under unloading.
Formula (60) may be used only for additional finite element of the second type as for additional finite element of the first type it is accepted that its strains $\Delta\varepsilon_{nonl,i} = 0$ (38).
The algorithm of the problem solving may be shorted by this formula.
In this case operations of the method of additional strains may be added. Scheme of this process for the allowance of the first nonlinear property is given at Figure 3.

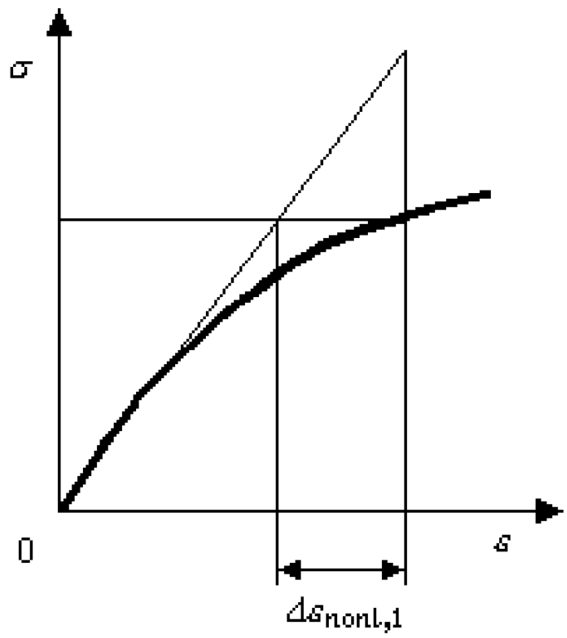

Figure 3. Scheme of the method of additional strains which is used by AFEM for the allowance of the first nonlinear property by means of AFE of the second type [19, 20].

Briefly the main characteristics of the additional finite elements of the first and the second type for the allowance of the any i-th nonlinear property are given in Table 1 [2, 20].
Formula (34) gives the main way for determination of additional load for taking into account of any i-th nonlinear property. Formulae (48) and (60) are suitable for two special cases.
Thus three ways for formation of additional load are possible.

6. FORMATION OF RIGHT HAND PART OF THE SET EQUATIONS OF AFEM

Two ways for changing of the right hand part of the set of equations (19) – (23) may be used for taking into account the i-th nonlinear property.
In the first case stiffness matrix of each additional design diagrams ΔK_i is formed on the base of stiffness matrices $\Delta K_{nonl,e,i}$ of correspon-

*Table 1. The main characteristics of additional finite elements for the allowance of the **i-th** nonlinear property.*

N	Type of characteristic	Finite element without the *i*-th nonlinear property	Finite element with the *i*-th nonlinear property	Additional finite element for the allowance of the *i*-th nonlinear property	
				Type 1	Type 2
1	Relationship between node reactions and displacements	$R_{nonl,i-1} = = K_{nonl,e,i-1} V$	$R_{nonl,i} = = K_{nonl,e,i} V$	$\Delta R_{nonl,i} = \Delta K_{nonl,e,i} V$	
2	Stiffness matrices	$K_{nonl,e,i-1}$	$K_{nonl,e,i}$	$\Delta K_{nonl,e,i} = K_{nonl,e,i} - K_{nonl,e,i-1}$	
3	Node reactions	$R_{nonl,i-1}$	$R_{nonl,i}$	$\Delta R_{nonl,i} = R_{nonl,i} - R_{nonl,i-1}$	
4	Relationship between node reactions and stresses	$R_{nonl,i-1} = = C\sigma_{nonl,i-1}$	$R_{nonl,i} = = C\,\sigma_{nonl,i}$	$\Delta R_{nonl,i} = = C\,\Delta\sigma_{nonl,i}$	–
5	Relationship between node reactions and strains	$R_{nonl,i-1} = = G\varepsilon_{nonl,i-1}$	$R_{nonl,i} = = G\varepsilon_{nonl,i}$	–	$\Delta R_{nonl,i} = G\Delta\varepsilon_{nonl,i}$
6	Stresses	$\sigma_{nonl,i-1}$	$\sigma_{nonl,i}$	$\Delta\sigma_{nonl,i} = = \sigma_{nonl,i} - \sigma_{nonl,i-}$	$\Delta\sigma_{nonl,i} = 0$
7	Strains	$\varepsilon_{nonl,i-1}$	$\varepsilon_{nonl,i}$	$\Delta\varepsilon_{nonl,i} = 0$	$\Delta\varepsilon_{nonl,i} = = \varepsilon_{nonl,i} - \varepsilon_{nonl,i-1}$
8	Additional load	$F_{e,i} = -\Delta R_{nonl,i}$	–	–	–

ding additional finite elements taking into account given nonlinear property.
The formula (27) is used for it only.
In the second case every component

$$(-\Delta K_i V)$$

is formed on the base of the values

$$(-\Delta R_{nonl,i})$$

for taking into account the result of action of corresponding additional finite element taking into account this nonlinear property.
Three formulae (33), (47) and (59) may be used for calculation of node reactions $\Delta R_{nonl,i}$.
The choice suitable formula depends on characteristics of considered nonlinear property.

Both of these ways must correspond to real conditions of nonlinear analysis at limit state and failure model.

7. CONCLUSIONS

Suggested by AFEM set of linear equations according to ideal failure model of structure is developed on basis of theory of FEM and theory of ultimate equilibrium. It allows:

1) to take into account each nonlinear property which is appeared at definite stage of operating period of structure;
2) to introduce the criterion of collapse structure according to logic of FEM;
3) to realize nonlinear analysis of structures at limit states according to requirements of Codes;
4) to introduce different theoretical models of behavior of structures according to degree of

reaching of ultimate limit state due to properties of additional finite elements;
5) to use of all advantages of method of elastic decisions (additional loads).

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