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ABOUT WAVELET-BASED MULTIGRID NUMERICAL METHOD OF STRUCTURAL ANALYSIS WITH THE USE OF DISCRETE HAAR BASIS

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Abstract: The distinctive paper is devoted to so-called multigrid (particularly two-grid) method of structural analysis based on discrete Haar basis (one-dimensional, two-dimensional and three-dimensional problems are under consideration). Approximations of the mesh functions in discrete Haar bases of zero and first levels are described (the mesh function is represented as the sum in which one term is its approximation of the first level, and the second term is so-called complement (up to the initial state) on the grid of the first level). Special projectors are constructed for the spaces of vector functions of the original grid to the space of their approximation on the first-level grid and its complement (the refinement component) to the initial state. Basic scheme of the two-grid method is presented. This method allows solution of boundary problems of structural mechanics with the use of matrix operators of significantly smaller dimension. It should be noted that discrete analogue of the initial operator equation is a system of linear algebraic equations which is constructed with the use of finite element method or finite difference method. Block Gauss method can be used for direct solution.

Keywords: structural analysis, wavelet analysis, numerical methods, multigrid methods, two-grid method, discrete Haar basis, reduction, local structural analysis, approximation, boundary problem

О ВЕЙВЛЕТ-РЕАЛИЗАЦИИ МНОГОСЕТОЧНОГО ЧИСЛЕННОГО МЕТОДА РАСЧЕТА КОНСТРУКЦИЙ НА ОСНОВЕ ДИСКРЕТНОГО БАЗИСА ХААРА

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Аннотация: Настоящая статья посвящена так называемому многосеточному (в частности, двухсеточному) методу расчета строительных конструкций, основанному на использовании дискретного базиса Хаара (рассматриваются одномерные, двумерные и трехмерные задачи). Описана аппроксимации сеточных функций на нулевом и первом уровнях дискретного базиса Хаара (сеточная функция представляется в виде суммы, в которой первое слагаемое соответствует аппроксимации на первом уровне, а второе слагаемое представляет собой так называемую детализацию (дополнение до исходного состояния) на сетке первого уровня. Строятся специальные проекторы пространства векторных функций на исходной стеке на пространство их аппроксимаций на сетке первого уровня и детализацию (дополнение до исходного состояния). Изложена общая схема двухсеточного метода. Данный метод позволяет решать краевые задачи строительной механики с использованием соответствующих матричных операторов существенно меньшей размерности. Следует отметить, что дискретным аналогом исходного операторного уравнения является система линейных алгебраических уравнений, формируемая с использованием техники метода конечных элементов или метода конечных разностей. Для прямого решения указанной системы может использоваться блочный метод Ланцоша.

Ключевые слова: расчеты строительных конструкций, вейвлет-анализ, численные методы, многосеточные методы, двухсеточный метод, дискретный базис Хаара, редукция, локальный расчет конструкций, аппроксимация, краевая задача

1. ONE-DIMENSIONAL PROBLEMS

1.1. About alternative discrete basis.

Let us consider the following functions:

– father wavelet (Figure 1),

$$\Phi(x)=\begin{cases}1, & 0\le x<1\\ 0, & x<0\vee x\ge 1;\end{cases}\qquad(1.1)$$

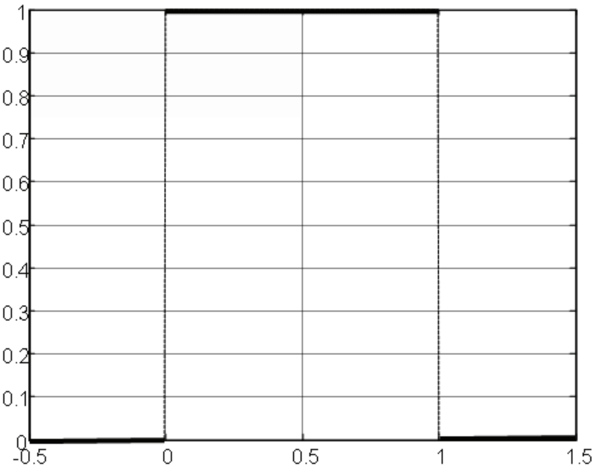


Figure 1. Father wavelet.

– the first mother wavelet (Figure 2),

$$\Psi_1(x)=\begin{cases}1, & 0\le x<1/3\\ -1, & 2/3\le x<1\\ 0, & x<0\vee 1/3\le x<2/3\vee x\ge 1;\end{cases}\qquad(1.2)$$

– the second mother wavelet (Figure 3) i.e. variant of French Hat,

$$\Psi_1(x)=\begin{cases}-1, & 0\le x<1/3\vee 2/3\le x<1\\ 2, & 1/3\le x<2/3\\ 0, & x<0\vee x\ge 1;\end{cases}\qquad(1.3)$$

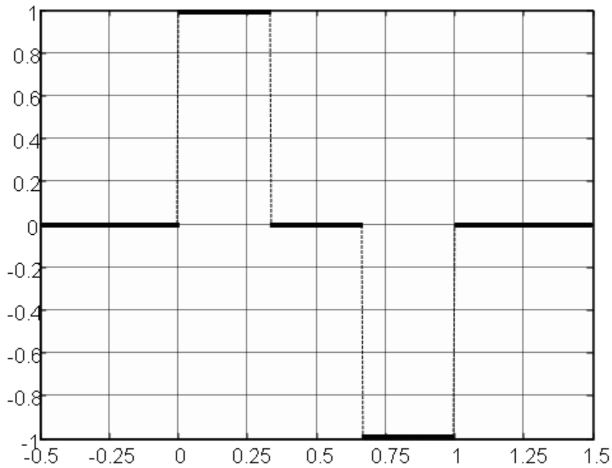


Figure 2. The first Mother wavelet.

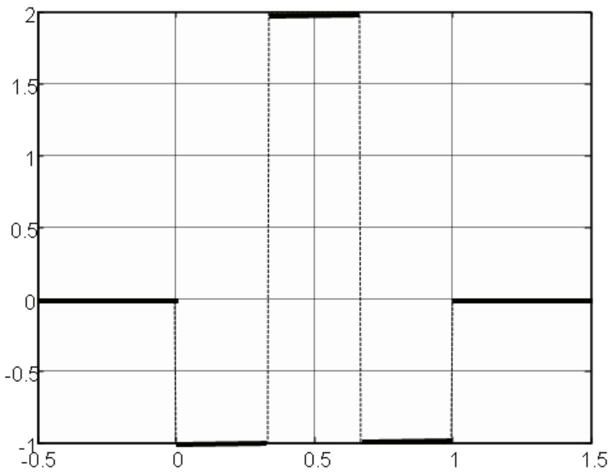


Figure 3. The second Mother wavelet.

Matrix of transition from one level to the next level has the same structure as in the Haar basis [1-25]

$$\begin{bmatrix}Q & & \\ & \ddots & \\ & & Q\end{bmatrix},$$

where for the Haar basis we have

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and for considering basis we have

$$Q = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & & \\ & 1/\sqrt{2} & \\ & & 1/\sqrt{6} \end{bmatrix}. \quad (1.4)$$

Let us denote a family of discrete functions:

$$\{\overline{\psi}_j^p\}, \quad j = 1, 2, \ldots, N_p, \quad p = 1, \ldots, M,$$

where M is the number of levels; $N = 3^M$ is space dimension,

$$\underline{1 \le p < M}: \quad \overline{\psi}_j^p = [\overline{\psi}_{1,j}^p \quad \overline{\psi}_{2,j}^p], \quad (1.5)$$

$$\psi_{1,j}^p(i) = \alpha_{1,p} \Psi_1\left(\frac{i - 0.5}{3^p} - (j-1) \right); \quad (1.6)$$

$$\psi_{2,j}^p(i) = \alpha_{2,p} \Psi_2\left(\frac{i - 0.5}{3^p} - (j-1) \right); \quad (1.7)$$

$$\underline{p = M}: \quad \overline{\psi}_1^M = [\overline{\psi}_{1,1}^M \quad \overline{\psi}_{2,1}^M \quad \overline{\phi}_{2,1}^M]; \quad (1.8)$$

$$\phi_1^M(i) = \alpha_M \Phi\left(\frac{i - 0.5}{3^M} \right); \quad (1.9)$$

$N_p = N / 3^p$ is the number of $\psi_{1,j}^p$ and $\psi_{2,j}^p$ at level number p;

$$\alpha_p = \frac{1}{\sqrt{3^p}}; \quad \alpha_{1,p} = \frac{1}{\sqrt{2^p}}; \quad \alpha_{2,p} = \frac{1}{\sqrt{6^p}} \quad (1.10)$$

are normalizing factors.

Because $1/3$ has no finite exact value, the expression

$$\frac{i - 0.5}{3^p}$$

is used to evaluate correctly instead of

$$\frac{i-1}{3^p}.$$

The set $\{\overline{\psi}_j^p\}$ presented in this way is a basis in the N-dimensional vector space. Moreover, this basis will be orthonormal.

Let's consider a given vector

$$\overline{u} \in R^N, \quad \overline{u} = \{u_i\}_{1 \le i \le N}.$$

We present its decomposition in the Haar basis

$$\overline{u} = \sum_{p=1}^{M} \left(\sum_{j=1}^{N_p} (v_j^p \overline{\psi 1}_j^p + w_j^p \overline{\psi 2}_j^p \right) + u_1^M \overline{\phi}_1^M, \quad (1.11)$$

where we have

$$v_j^p = (\overline{\psi}_{1,j}^p, \overline{u}); \quad w_j^p = (\overline{\psi}_{2,j}^p, \overline{u});$$
$$u_1^M = (\overline{\phi}_1^M, \overline{u}), \quad j = 1, 2, \ldots, N_p, \quad p = 1, \ldots, M. \quad (1.12)$$

1.2. Determination of averaging coefficients. Let us consider an example of a grid partition for $N = 9 = 3^2$, i.e. $M = 2$. Then the location of nodes by levels can be represented as shown in Figure 4. The formation of the decomposition coefficients at the first level is carried out according to the following formulas:

$$k = 1, 2, 3$$

$$u_k^1 = \frac{1}{\sqrt{3}} (u_{3k-2}^0 + u_{3k-1}^0 + u_{3k}^0) = \sqrt{3} \tilde{u}_k^0; \quad (1.13)$$

$$\tilde{u}_k^0 = \frac{1}{3} (u_{3k-2}^0 + u_{3k-1}^0 + u_{3k}^0); \quad (1.14)$$

$$\begin{aligned} v_k^1 &= \frac{1}{\sqrt{2}} (u_{3k-2}^0 - u_{3k}^0) = \\ &= -\frac{1}{\sqrt{2}} 2h \frac{u_{3k}^0 - u_{3k-2}^0}{2h} = -\sqrt{2} h (D_0 u^0)_{3k-1}; \end{aligned} \quad (1.15)$$

$$\begin{aligned} w_k^1 &= \frac{1}{\sqrt{6}} (-u_{3k-2}^0 + 2u_{3k-1}^0 - u_{3k}^0) = \\ &= -\frac{1}{\sqrt{6}} h^2 \frac{u_{3k-2}^0 - 2u_{3k-1}^0 + u_{3k}^0}{h^2}) = \\ &= -\frac{h^2}{\sqrt{6}} (D^2 u^0)_{3k-1}. \end{aligned} \quad (1.16)$$

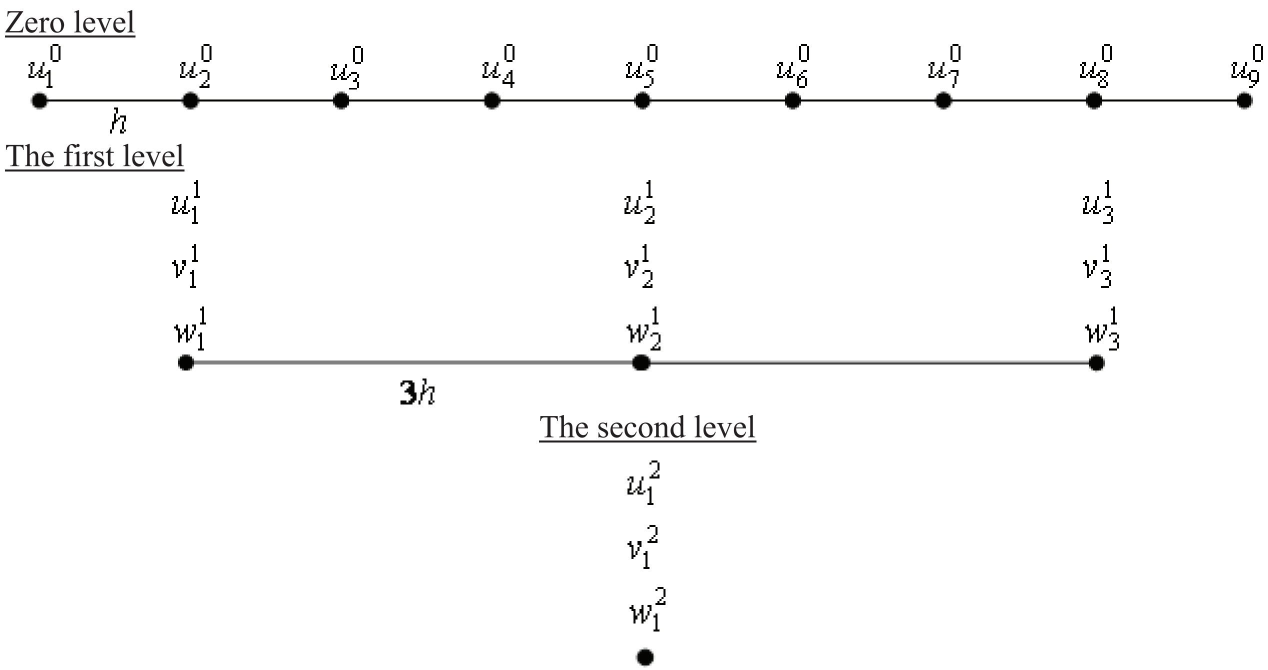


Figure 4. About determination of averaging coefficients.

The formation of the decomposition coefficients at the second level is carried out according to the following formulas:

$$u_k^2 = \frac{1}{\sqrt{3}}(u_1^1 + u_2^1 + u_3^1)\,; \tag{1.17}$$

$$\begin{aligned} v_1^2 &= \frac{1}{\sqrt{2}}(u_1^1 - u_3^1) = -\frac{1}{\sqrt{2}}6h\frac{u_3^1 - u_1^1}{6h} = \\ &= -\frac{\sqrt{3}}{\sqrt{2}}6h\frac{\tilde{u}_3^0 - \tilde{u}_1^0}{6h} = -\frac{\sqrt{3}}{\sqrt{2}}6h(D_0\tilde{u}^0)_2\,; \end{aligned} \tag{1.18}$$

$$\begin{aligned} w_1^1 &= \frac{1}{\sqrt{6}}(-u_1^1 + 2u_2^1 - u_3^1) = \\ &= -\frac{1}{\sqrt{6}}(3h)^2\frac{u_1^1 - 2u_2^1 + u_3^1}{(3h)^2}) = \\ &= -\frac{\sqrt{3}}{\sqrt{6}}(3h)^2\frac{\tilde{u}_1^0 - 2\tilde{u}_2^0 + \tilde{u}_3^0}{(3h)^2}) = \\ &= -\frac{(3h)^2}{\sqrt{2}}(D^2\tilde{u}^0)_2\,. \end{aligned} \tag{1.19}$$

For averaging (reduction) we assume

$$\begin{aligned} &(D_0u^0)_{3k-1} = (D_0\tilde{u}^0)_2\,, \\ &\qquad (D^2u^0)_{3k-1} = (D^2\tilde{u}^0)_2\,, \quad k = 1,2,3\,. \end{aligned} \tag{1.20}$$

The following dependencies can be noted

$$\begin{cases} (D_0u^0)_{3k-1} = -\dfrac{1}{\sqrt{2}h}v_k^1 \\ (D^2u^0)_{3k-1} = -\dfrac{\sqrt{6}}{h^2}w_k^1\,; \end{cases} \tag{1.21}$$

$$\begin{cases} (D_0\tilde{u}^0)_2 = -\dfrac{\sqrt{2}}{\sqrt{3}\cdot 6h}v_1^2 \\ (D^2\tilde{u}^0)_2 = -\dfrac{\sqrt{2}}{(3h)^2}w_1^2\,; \end{cases} \tag{1.22}$$

and then we have

$$\begin{cases} v_k^1 = \dfrac{1}{3\sqrt{3}}v_1^2 \\ w_k^1 = \dfrac{1}{3^2\sqrt{3}}w_1^2\,, \end{cases} \quad k = 1,2,3\,. \tag{1.23}$$

1.3. The final formulas.

Let the initial one-dimensional domain be given as interval with length L. We can use simple mesh (grid) for approximation of domain and divide interval (the initial domain) into $(N-1)$ equal parts. We can represent the mesh vector function $\overline{u} = [u_1 \ \ldots \ u_N]^T$ in the form

$$\overline{u} = \sum_{j=1}^{N_0} u_j^0 \overline{\Phi}_j^0 , \tag{1.24}$$

where we have

$$N_0 = N = 3^M ; \tag{1.25}$$

$$u_j^0 = u_j , \quad 1 \le j \le N ; \tag{1.26}$$

$$\overline{\Phi}_j^0(i) = \begin{cases} 1, & i = j \\ 0, & i \ne j \end{cases} \tag{1.27}$$

– j -th vector of unit basis of level 0,

$$1 \le j \le N_0 = N , \quad 1 \le i \le N .$$

At the same time, this grid function can be represented in the basis of the first level as

$$\overline{u} = \sum_{j=1}^{N_1} u_j^1 \overline{\Phi}_j^1 + \sum_{j=1}^{N_1} v_j^1 \overline{\Psi}_{1,j}^1 + \sum_{j=1}^{N_1} w_j^1 \overline{\Psi}_{2,j}^1 , \tag{1.28}$$

where we have

$$N_1 = N/3 ;$$

$$u_j^1 = (\overline{u}, \overline{\Phi}_j^1) ; \quad v_j^1 = (\overline{u}, \overline{\Psi 1}_j^1) ;$$

$$w_j^1 = (\overline{u}, \overline{\Psi}_{2,j}^1) , \quad 1 \le j \le N_1 , \tag{1.29}$$

$$\overline{u}^1 = [u_1^1 \ldots u_{N_1}^1]^{\mathrm{T}} , \quad \overline{v}^1 = [v_1^1 \ldots v_{N_1}^1]^{\mathrm{T}} ,$$

$$\overline{w}^1 = [w_1^1 \ldots w_{N_1}^1]^{\mathrm{T}} . \tag{1.30}$$

$$\overline{\Phi}_j^1 = \alpha(\overline{\Phi}_{3j-2}^0 + \overline{\Phi}_{3j-1}^0 + \overline{\Phi}_{3j}^0) \tag{1.31}$$

is the j -th approximation vector of discrete wavelet basis of the first level, $1 \le j \le N_1$;

$$\overline{\Psi}_{1,j}^1 = \alpha_1(\overline{\Phi}_{3j-2}^0 - \overline{\Phi}_{3j}^0) \tag{1.32}$$

is the j -th refinement vector of discrete wavelet basis of the first level, $1 \le j \le N_1$;

$$\overline{\Psi}_{2,j}^1 = \alpha_2(-\overline{\Phi}_{3j-2}^0 + 2\overline{\Phi}_{3j-1}^0 - \overline{\Phi}_{3j}^0) \tag{1.33}$$

is the j -th refinement vector of discrete wavelet basis of the first level, $1 \le j \le N_1$;

$$\begin{cases} \alpha = 1/\sqrt{3} \\ \alpha_1 = 1/\sqrt{2} \\ \alpha_2 = 1/\sqrt{6} \end{cases} \tag{1.34}$$

are normalizing factors.

Based on the representation of (1.28), the mesh function is represented as a sum, where the first term is its approximation on the mesh (grid) of the first level, consisting of N_1 nodes, and the second and third terms are called refinement (addition to the initial state) on the mesh (grid) of the first level.

The representation (1.28) can be written as

$$\overline{u} = \overline{u}_1^0 + \overline{v}_1^0 + \overline{w}_1^0 ; \tag{1.35}$$

$$\overline{u}_1^0 = \sum_{j=1}^{N_1} u_j^1 \overline{\Phi}_j^1 = \Phi_1 \Phi_1^T \overline{u} = \Phi_1 \overline{u}^1 ; \tag{1.36}$$

$$\overline{v}_1^0 = \sum_{j=1}^{N_1} v_j^1 \overline{\Psi}_{1,j}^1 = \Psi_{1,1} \Psi_{1,1}^T \overline{u} = \Psi_{1,1} \overline{v}^1 ; \tag{1.37}$$

$$\overline{w}_1^0 = \sum_{j=1}^{N_1} w_j^1 \overline{\Psi}_{2,j}^1 = \Psi_{2,1} \Psi_{2,1}^T \overline{u} = \Psi_{2,1} \overline{w}^1 , \tag{1.38}$$

where we have

$$\Phi_1 = [\, \overline{\Phi}_1^1 \ \ldots \ \overline{\Phi}_{N_1}^1 \,] ; \tag{1.39}$$

$$\Psi_{1,1} = [\overline{\Psi}_{1,1}^1 \ \ldots \ \overline{\Psi}_{1,N_1}^1] ; \tag{1.40}$$

$$\Psi_{2,1} = [\overline{\Psi}_{2,1}^1 \ \ldots \ \overline{\Psi}_{2,N_1}^1] \tag{1.41}$$

matrices of size $N \times N_1$, the columns of which are respectively approximation and refinement vectors of the discrete basis of the first level.

Due to the orthonormality of the Haar basis [1-7,9-25], the operators

$$P_\Phi = \Phi_1 \Phi_1^T , \quad P_{\Psi,1} = \Psi_{1,1} \Psi_{1,1}^T , \quad P_{\Psi,2} = \Psi_{2,1} \Psi_{2,1}^T \tag{1.42}$$

are projectors of the space of vector functions of the original mesh (grid) to the space of their approximation on the first-level grid and its complement (the refining component) to the initial state, respectively.
spectively.

1.4. Basic scheme of the two-grid method.
Let systems of linear algebraic equations

$$A\bar{u} = \bar{f} \qquad (1.43)$$

are discrete analogs of some operator equation defined on a given interval of order N.
We can substitute in (1.43) the expression for $\bar{u}$ in the form (1.35). Then we can multiply, in turn, both sides of the equality on the left by the matrices Φ_1^T and $\Psi_{1,1}^T, \Psi_{2,1}^T$. Thus we have

$$\begin{aligned} &\Phi_1^T A\Phi_1\bar{u}^1 + \Phi_1^T A\Psi_{1,1}\bar{v}^1 + \\ &\qquad + \Phi_1^T A\Psi_{2,1}\bar{w}^1 = \Phi_1^T \bar{f} \\ &\Psi_{1,1}^T A\Phi_1\bar{u}^1 + \Psi_{1,1}^T A\Psi_{1,1}\bar{v}^1 + \\ &\qquad + \Psi_{1,1}^T A\Psi_{2,1}\bar{w}^1 = \Psi_{1,1}^T \bar{f} \\ &\Psi_{2,1}^T A\Phi_1\bar{u}^1 + \Psi_{2,1}^T A\Psi_{1,1}\bar{v}^1 + \\ &\qquad + \Psi_{2,1}^T A\Psi_{2,1}\bar{w}^1 = \Psi_{2,1}^T \bar{f}; \end{aligned} \qquad (1.44)$$

or

$$\begin{aligned} A_{1,1}\bar{u}^1 + A_{1,2}\bar{v}^1 + A_{1,3}\bar{w}^1 &= \bar{f}_u \\ A_{2,1}\bar{u}^1 + A_{2,2}\bar{v}^1 + A_{2,3}\bar{w}^1 &= \bar{f}_v \\ A_{3,1}\bar{u}^1 + A_{3,2}\bar{v}^1 + A_{3,3}\bar{w}^1 &= \bar{f}_w, \end{aligned} \qquad (1.45)$$

where we have

$$A_{1,1} = \Phi_1^T A\Phi_1; \quad A_{1,2} = \Phi_1^T A\Psi_{1,1}; \qquad (1.46)$$

$$A_{1,3} = \Phi_1^T A\Psi_{2,1}; \quad A_{2,1} = \Psi_{1,1}^T A\Phi_1; \qquad (1.47)$$

$$A_{2,2} = \Psi_{1,1}^T A\Psi_{1,1}; \quad A_{2,3} = \Psi_{1,1}^T A\Psi_{2,1}; \qquad (1.48)$$

$$A_{3,1} = \Psi_{2,1}^T A\Phi_1; \quad A_{3,2} = \Psi_{2,1}^T A\Psi_{1,1}; \qquad (1.49)$$

$$A_{3,3} = \Psi_{2,1}^T A\Psi_{2,1} \qquad (1.50)$$

are block matrices of size $N_1 \times N_1$.

$$\bar{f}_u = \Phi_1^T \bar{f}; \quad \bar{f}_v = \Psi_{1,1}^T \bar{f}; \quad \bar{f}_w = \Psi_{2,1}^T \bar{f} \qquad (1.51)$$

are right-side vectors of length N_1.
We can find the solution of the system (1.45) using the block Gaussian method.
The expanded block matrix has the form:

$$\left[\begin{array}{ccc|c} A_{1,1} & A_{1,2} & A_{1,3} & \bar{f}_u \\ A_{2,1} & A_{2,2} & A_{2,3} & \bar{f}_v \\ A_{3,1} & A_{3,2} & A_{3,3} & \bar{f}_w \end{array}\right]. \qquad (1.52)$$

We have the following result of forward algorithm:
– the first step

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & \bar{f}_u \\ 0 & A_{22}^1 & A_{23}^1 & \bar{f}_v^1 \\ 0 & A_{32}^1 & A_{33}^1 & \bar{f}_w^1 \end{array}\right], \qquad (1.53)$$

where we have

$$A_{2,2}^1 = A_{2,2} - C_{2,1}A_{1,2}; \quad A_{2,3}^1 = A_{2,3} - C_{2,1}A_{1,3}; \qquad (1.54)$$

$$\bar{f}_v^1 = \bar{f}_v - C_{2,1}\bar{f}_u; \quad C_{2,1} = A_{2,1}A_{1,1}^{-1}; \qquad (1.55)$$

$$A_{3,2}^1 = A_{3,2} - C_{3,1}A_{1,2}; \quad A_{3,3}^1 = A_{3,3} - C_{3,1}A_{1,3}; \qquad (1.56)$$

$$\bar{f}_w^1 = \bar{f}_w - C_{3,1}\bar{f}_u, \quad C_{3,1} = A_{3,1}A_{1,1}^{-1}. \qquad (1.57)$$

– the second step

$$\left[\begin{array}{ccc|c} A_{1,1} & A_{1,2} & A_{1,3} & \bar{f}_u \\ 0 & A_{2,2}^1 & A_{2,3}^1 & \bar{f}_v^1 \\ 0 & 0 & A_{3,3}^2 & \bar{f}_w^2 \end{array}\right], \qquad (1.58)$$

where we have

$$A_{3,3}^2 = A_{3,3}^1 - C_{3,2}A_{2,3}^2; \qquad (1.59)$$

$$\bar{f}_w^2 = \bar{f}_w^1 - C_{3,2}\bar{f}_v^1, \quad C_{3,2} = A_{3,2}^1(A_{2,2}^1)^{-1}. \qquad (1.60)$$

We have the following result of backward algorithm

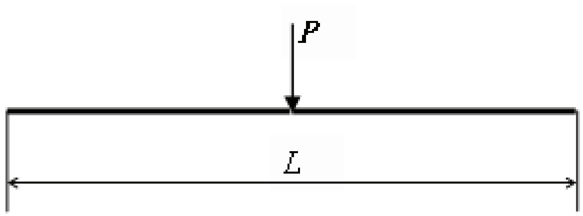


Figure 5. To the formulation of the boundary problem.

Figure 6. Discrete approximation of the domain.

$$\begin{aligned} \bar{w}^1 &= (A_{3,3}^2)^{-1} \bar{f}_w^2 \\ \bar{v}^1 &= (A_{2,2}^1)^{-1} (\bar{f}_v^1 - A_{2,3}^1 \bar{v}^1) \\ \bar{u}^1 &= A_{1,1}^{-1} (\bar{f}_u - A_{1,2} \bar{v}^1 - A_{1,3} \bar{w}^1). \end{aligned} \tag{1.61}$$

In accordance with formulas (1.35)-(1.38) solution of the considering problem (1.43) has the form

$$\bar{u} = \Phi_1 \bar{u}^1 + \Psi_{1,1} \bar{v}^1 + \Psi_{2,1} \bar{w}^1. \tag{1.62}$$

It should be noted that the proposed method makes it possible to obtain a solution with the use of matrices of size $N_1 = N/3$ and vectors of size $N_1 = N/3$.

1.5. Numerical sample.

Let us consider (as a model example) the numerical solution of the boundary problem of the bending of a beam on an elastic foundation with the following parameters: L=8 m is the length of the beam; h=1.3 m is the height of the cross section of the beam; b=1 m is the width of the cross section of the beam; $E = 2560 \cdot 10^4$ kN/m^2 is the modulus of elasticity; $P = 100$ kN is the load set at the middle point, $P_h = 100/h_b$; $k = 75 \cdot 10^3$ kN/m^3 is the coefficient characterizing the resistance of the foundation in the model of Winkler; $\bar{k} = k \cdot b$, $J = bh^3/12$ (Figure 5).
The definition of the deflection of the Bernoulli beam is reduced to the solution of the following boundary problem:

$$y^{(4)}(x) + 4\alpha^4 y(x) = F(x), \quad 0 < x < L; \tag{1.63}$$

$$\begin{cases} y(0) = y''(0) = 0 \\ y(L) = y''(L) = 0, \end{cases} \tag{1.64}$$

where we have

$$4\alpha^4 = \frac{\bar{k}}{EJ}; \quad F = \frac{P}{EJ} \delta(x - \frac{L}{2}). \tag{1.65}$$

Let us divide the interval (0, L) into equal parts with step (Figure 6).
If n is the total number of points, it is obvious that

$$h_b = L/(n-1). \tag{1.66}$$

Next, we move from the solution of the boundary problem (1.63)-(1.64) to the solution of the following system of difference equations:

$$\begin{aligned} & i = 1: \; y_1 = f_1 \\ & i = 2: \; -2y_1 + (5 + 4h_b^4 \alpha^4) y_2 - 4y_3 + y_4 = f_2 \\ & 2 < i < n-1: \; y_{i-2} - 4y_{i-1} + (6 + 4h_b^4 \alpha^4) y_i - \\ & \qquad - 4y_{i+1} + y_{i+2} = f_i \\ & i = n-1: \; y_{n-3} - 4y_{n-2} + (5 + 4h_b^4 \alpha^4) y_{n-1} - \\ & \qquad - 2y_n = f_{n-1} \\ & i = n: \; y_n = f_n, \end{aligned} \tag{1.67}$$

where we have

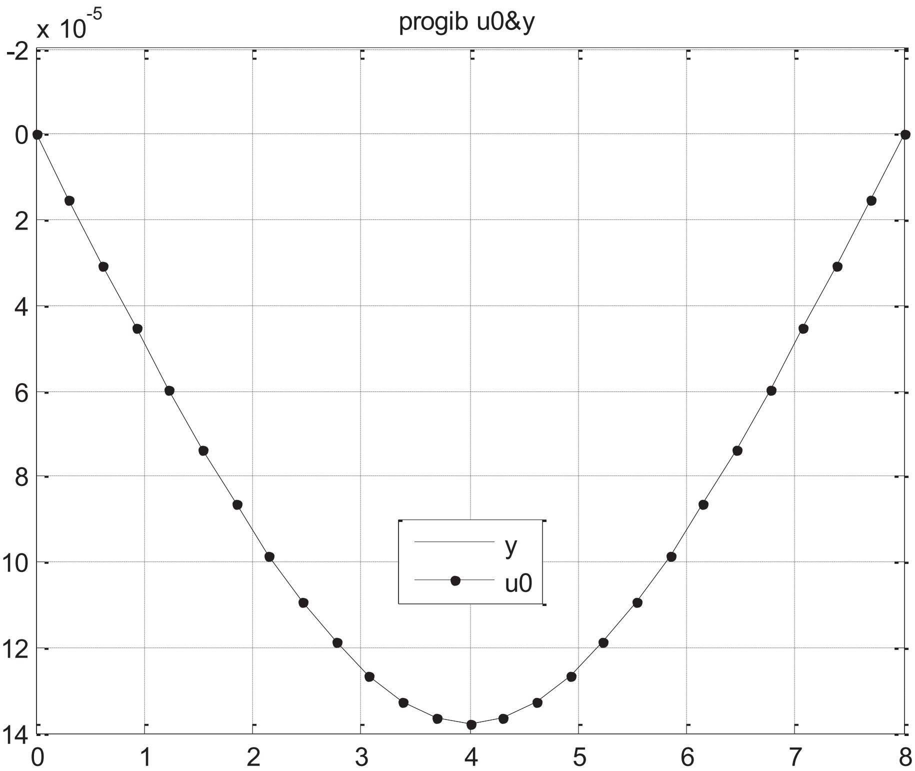


Figure 7. Comparison of results.

$$x_i = h_b(i-1)\,; \quad y_i = y(x_i)\,; \tag{1.68}$$

$$f_i = \begin{cases} 0, & i=1 \\ h_b^4[F(x_i)/h_b], & 2 \le i \le n-1 \\ 0, & i=n. \end{cases} \tag{1.69}$$

In accordance with the results of analysis comparative graphs of deflections were obtained. Direct solution of the system (1.67) and solution defined by the formula (1.62) are presented at Figure 7. It should be noted that full match results obtained.

2. TWO-DIMENSIONAL PROBLEMS

2.1. General information.

Let the initial two-dimensional domain be given as a rectangle [8]. Let L_1 and L_2 be lengths of sides of this rectangle in directions corresponding to Cartesian coordinates x_1 and x_2. We can use simple rectangular mesh (grid) for approximation of domain and divide each side of the rectangle (the initial domain) into $(N-1)$ equal parts. The corresponding mesh (grid) width are defined by formulas

$$h_1 = L_1/(N-1)\,; \quad h_2 = L_2/(N-1)\,. \tag{2.1}$$

Thus, the resulting mesh contains N^2 nodes.
Let us introduce the mesh function

$$\bar{u} = [u_{1,1}\ \ldots\ u_{1,N}\ \ldots\ u_{N,1}\ \ldots u_{N,N}]^T\,. \tag{2.2}$$

We can represent the mesh function (1.2) in the form

$$\bar{u} = \sum_{j_2=1}^{N_0}\sum_{j_1=1}^{N_0} u^0_{j_2,j_1}\,\overline{\Phi}^0_{j_2,j_1}\,, \tag{2.3}$$

where we have

$$N_0 = N\,; \tag{2.4}$$

$$u^0_{j_2,j_1} = u_{j_2,j_1}\,, \quad 1 \le j_1, j_2 \le N\,; \tag{2.5}$$

$$\overline{\Phi}^0_{j_2,j_1}(i_2,i_1)=\begin{cases}1, & i_1=j_1\wedge i_2=j_2\\ 0, & \neg(i_1=j_1\wedge i_2=j_2)\end{cases}$$

$$1\le j_2,j_1\le N_0=N\,,\ 1\le i_2,i_1\le N\,, \quad (2.6)$$

where $\overline{\Phi}^0_{j_2,j_1}$ is (j_2,j_1)-th vector of a unit basis or a discrete zero-level Haar basis

The mesh function (2) can also be represented in the form of an expansion in the Haar basis of the first level:

$$\overline{u}=\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1}u^1_{j_2,j_1}\overline{\Phi}^1_{j_2,j_1}+\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1}v^1_{1,j_2,j_1}\overline{\Psi}^1_{1,j_2,j_1}+ \\ +\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1}v^1_{2,j_2,j_1}\overline{\Psi}^1_{2,j_2,j_1}+\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1}v^1_{3,j_2,j_1}\overline{\Psi}^1_{3,j_2,j_1}, \quad (2.7)$$

where we have

$$N_1=N/2; \quad (2.8)$$

$$u^1_{j_2,j_1}=(\overline{u},\overline{\Phi}^1_{j_2,j_1});\quad v^1_{1,j_2,j_1}=(\overline{u},\overline{\Psi}^1_{1,j_2,j_1}), \quad (2.9)$$

$$v^1_{2,j_2,j_1}=(\overline{u},\overline{\Psi}^1_{2,j_2,j_1});\quad v^1_{3,j_2,j_1}=(\overline{u},\overline{\Psi}^1_{3,j_2,j_1}); \quad (2.10)$$

$$\overline{u}^1=[u^1_{1,1}\dots u^1_{1,N_1}\dots u^1_{N_1,1}\dots u^1_{N_1,N_1}]^T; \quad (2.11)$$

$$\overline{v}^1_k=[v^1_{k,1,1}\dots v^1_{k,1,N_1},\dots v^1_{k,N_1,1}\dots {}^1_{k,N_1,N_1}]^T,\ k=1,2,3; \quad (2.12)$$

$$\overline{\Phi}^1_{j_2,j_1}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1}+\overline{\Phi}^0_{2j_2-1,2j_1}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1}+\overline{\Phi}^0_{2j_2,2j_1}) \quad (2.13)$$

– (j_2,j_1)- th approximating vector of the discrete Haar basis of the first level, $1\le j_2,j_1\le N_1$;

$$\overline{\Psi}^1_{1,j_2,j_1}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1}-\overline{\Phi}^0_{2j_2-1,2j_1}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1}-\overline{\Phi}^0_{2j_2,2j_1}) \quad (2.14)$$

– $(1,j_2,j_1)$- th refining vector of the discrete Haar basis of the first level,

$$\overline{\Psi}^1_{2,j_2,j_1}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1}+\overline{\Phi}^0_{2j_2-1,2j_1}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1}-\overline{\Phi}^0_{2j_2,2j_1}); \quad (2.15)$$

– $(2,j_2,j_1)$-th refining vector of the discrete Haar basis of the first level,

$$\overline{\Psi}^1_{3,j_2,j_1}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1}-\overline{\Phi}^0_{2j_2-1,2j_1}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1}+\overline{\Phi}^0_{2j_2,2j_1}) \quad (2.16)$$

– $(3,j_2,j_1)$- th refining vector of the discrete Haar basis of the first level,

$$1\le j_2,j_1\le N_1; \quad (2.17)$$

$\alpha=1/2$ is normalizing factor.

In accordance with (2.7), the mesh function $\overline{u}$ is represented as a sum in which the first summand (the first sum) is its approximation on the grid of the first level including N_1^2 nodes, and other three terms are called the refinement (the complement to the initial state) on the grid the first level. The representation (2.7) can be written in the form

$$\overline{u}=\overline{u}^0_1+\overline{v}^0_{1,1}+\overline{v}^0_{2,1}+\overline{v}^0_{3,1}; \quad (2.18)$$

$$\overline{u}^0_1=\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1}u^1_{j_2,j_1}\overline{\Phi}^1_{j_2,j_1}=\Phi_1\Phi_1^T\overline{u}=\Phi_1\overline{u}^1; \quad (2.19)$$

$$\overline{v}^0_{k,1}=\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1}v^1_{k,j_2,j_1}\overline{\Psi}^1_{k,j_2,j_1}=\Psi_{k,1}\Psi^T_{k,1}\overline{u}=\Psi_{k,1}\overline{v}^1_k, \\ k=1,2,3, \quad (2.20)$$

where we have

$$\Phi_1=[\overline{\Phi}^1_{1,1}\dots\overline{\Phi}^1_{1,N_1},..\overline{\Phi}^1_{N_1,1}\dots\overline{\Phi}^1_{N_1,N_1}]; \quad (2.21)$$

$$\Psi_{k,1}=[\overline{\Psi}^1_{k,1,1}\dots\overline{\Psi}^1_{k,1,N_1}\dots\overline{\Psi}^1_{k,N_1,1}\dots\overline{\Psi}^1_{k,N_1,N_1}], \\ k=1,2,3 \quad (2.22)$$

are matrices of size $N^2\times N_1^2$, whose columns are, respectively, approximating and detailing vectors of the discrete Haar basis of the first level.

Due to the orthonormality of the Haar basis [1-7,9-25], the operators

$$P_\Phi=\Phi_1\Phi_1^T,\ P_{\Psi,k}=\Psi_{k,1}\Psi^T_{k,1},\ k=1,2,3 \quad (2.23)$$

are projectors of the space of vector functions of the original grid to the space of their approximation on the first-level grid and its complement (the refining component) to the initial state, respectively.

2.2. Basic scheme of the two-grid method.
Let systems of linear algebraic equations

$$A\bar{u} = \bar{f} \tag{2.24}$$

are discrete analogs of some operator equation defined on a given rectangular prism of order N^2.
We can substitute in (2.24) the expression for $\bar{u}$ in the form (2.18). Then we can multiply, in turn, both sides of the equality on the left by the matrices Φ_1^T and $\Psi_{k,1}^T$, $k = 1,2,3$. Thus we have

$$\begin{aligned}
&\Phi_1^T A\Phi_1\bar{u}^1 + \sum_{k=1}^{3}\Phi_1^T A\Psi_{k1}\bar{v}_k^1 = \Phi_1^T \bar{f}\\
&\Psi_{1,1}^T A\Phi_1\bar{u}^1 + \sum_{k=1}^{3}\Psi_{1,1}^T A\Psi_{k,1}\bar{v}_k^1 = \Psi_{1,1}^T \bar{f}\\
&\Psi_{2,1}^T A\Phi_1\bar{u}^1 + \sum_{k=1}^{3}\Psi_{2,1}^T A\Psi_{k,1}\bar{v}_k^1 = \Psi_{2,1}^T \bar{f}\\
&\Psi_{3,1}^T A\Phi_1\bar{u}^1 + \sum_{k=1}^{3}\Psi_{3,1}^T A\Psi_{k,1}\bar{v}_k^1 = \Psi_{3,1}^T \bar{f}
\end{aligned} \tag{2.25}$$

Equations (2.25) can be rewritten in the form

$$\sum_{i=1}^{4} A_{i,j}\bar{w}_j = \bar{f}_i, \quad i = 1, \ldots, 4 . \tag{2.26}$$

where $A_{i,j}$ are block matrices of size $N_1^2 \times N_1^2$; $\bar{f}_i$ and $\bar{w}_j$ are block vectors of size N_1^2,

$$A_{1,1} = \Phi_1^T A\Phi_1; \quad A_{1,k+1} = \Phi_1^T A\Psi_{k,1},$$

$$A_{k+1,1} = \Psi_{k,1}^T A\Phi_1, \quad k = 1,2,3; \tag{2.27}$$

$$A_{k+1,p+1} = \Psi_{k,1}^T A\Psi_{p+1}, \quad k,p = 1,2,3; \tag{2.28}$$

$$\bar{f}_1 = \Phi_1^T \bar{f}, \quad \bar{f}_{k+1} = \Psi_{k,1}^T \bar{f}, \quad k = 1,\ldots,7; \tag{2.29}$$

$$\bar{w}_1 = \bar{u}^1, \quad \bar{w}_{k+1} = v_k^1, \quad k = 1,\ldots,7. \tag{2.30}$$

We can find the solution of the system (2.26) using the block Gaussian method.
The expanded block matrix has the form:

$$\left[\begin{array}{cccc|c}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \bar{f}_1\\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & \bar{f}_2\\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & \bar{f}_3\\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & \bar{f}_4
\end{array}\right]. \tag{2.31}$$

We have the following result of forward algorithm:

$$\left[\begin{array}{cccc|c}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \bar{f}_1\\
0 & A_{2,2}^1 & A_{2,3}^1 & A_{2,4}^1 & \bar{f}_2^1\\
0 & 0 & A_{3,3}^2 & A_{3,4}^2 & \bar{f}_3^2\\
0 & 0 & 0 & A_{4,4}^3 & \bar{f}_4^3
\end{array}\right], \tag{2.32}$$

where we have

$$A_{i,j}^k = A_{i,j}^{k-1} - C_{i,k}A_{k,j}^{k-1}, \quad k = 1,\ldots,4,$$
$$i = k+1,\ldots,4, \quad j = k+1,\ldots,4; \tag{2.33}$$

$$\bar{f}_i^k = \bar{f}_i^{k-1} - C_{i,k}\bar{f}_k^{k-1}, \quad k = 1,\ldots,4,$$
$$i = k+1,\ldots,4; \tag{2.34}$$

$$C_{i,k} = A_{i,k}^{k-1}(A_{k,k}^{k-1})^{-1}, \quad k = 1,\ldots,3,$$
$$i = k+1,\ldots,4. \tag{2.35}$$

We have the following result of backward algorithm

$$\bar{w}_8 = (A_{4,4}^3)^{-1}\bar{f}_4^3; \tag{2.36}$$

$$\bar{w}_i = (A_{i,i}^{i-1})^{-1}\left(\bar{f}_i^{i-1} - \sum_{j=i+1}^{4} A_{i,i}^{i-1}\bar{w}_j\right), \ i = 3,2,1. \tag{2.37}$$

In accordance with formulas (2.18)-(2.20) solution of the considering problem (2.24) has the form

$$\begin{aligned}
\bar{u} &= \bar{u}_1^0 + \sum_{k=1}^{7}\bar{v}_{k,1}^0 =\\
&= \Phi_1\bar{u}^1 + \sum_{k=1}^{7}\Psi_{k,1}\bar{v}_k^1 = \Phi_1\bar{w}_1 + \sum_{k=1}^{7}\Psi_{k,1}\bar{w}_{k+1}.
\end{aligned} \tag{2.38}$$

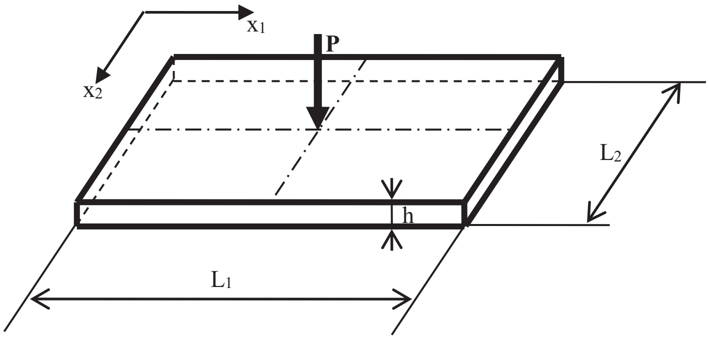


Figure 8. To the formulation of the boundary problem.

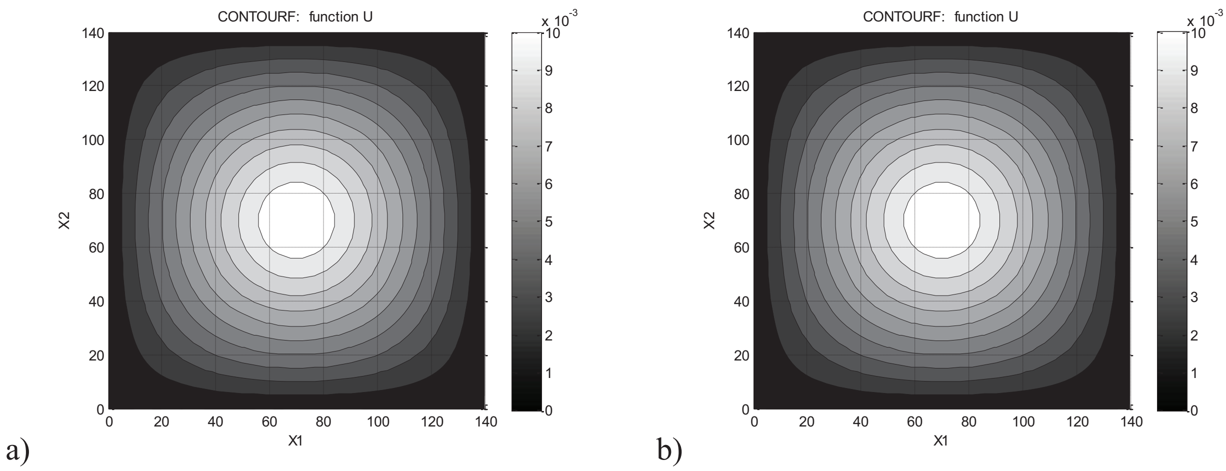


Figure 9. Solution of the boundary problem: a) solution of problem in initial formulation; b) solution with the use of two-grid method.

It should be noted that the proposed method makes it possible to obtain a solution with the use of matrices of size $N_1^2 \times N_1^2$ and vectors of size N_1^2 ($N_1^2 = (N/2)^2$).

3.3. Numerical samples.

Let us consider (as an example) the definition of bending a rectangular plate, hinged on the edges, loaded in the center by the concentrated force (Figure 8).

Design scheme of a rectangular plate is shown on Figure 8: $L_1 = L_2 = 140$ mm are dimensions of plate; $h = 20$ cm is thickness of plate; $E = 3 \cdot 10^3$ kN/cm² is modulus of elasticity of the material of plate; $\nu = 0.16$ is Poisson's ratio of the material of plate; $P = 100$ kN is parameter of concentrated load.

We can use finite difference method for numerical solution of the considering problem. Thus, we can get resultant system of equations with respect to the nodal deflections,

$$A\overline{U} = \bar{f}, \tag{2.39}$$

where A is matrix of coefficients; $\bar{f}$ is load vector; $\overline{U}$ is vector of unknown nodal deflections having the following structure

$$\overline{U} = [U_{1,1} \ \dots \ U_{1,N} \ \dots \ U_{N,1} \dots U_{N,N}]^T; \tag{2.40}$$

$U_{i,j} = u(x_{1,j}, x_{2,i})$ is the deflection in the node (i, j), $i, j = 1, \dots, N$.

In accordance with the results of analysis graphs of deflections were obtained. Direct solution of the system (2.39) and solution defined by the formula (2.38) are presented at Figure 9.

Let us consider (as an example) the static analysis of a trapezoidal beam-wall (Figure 10).

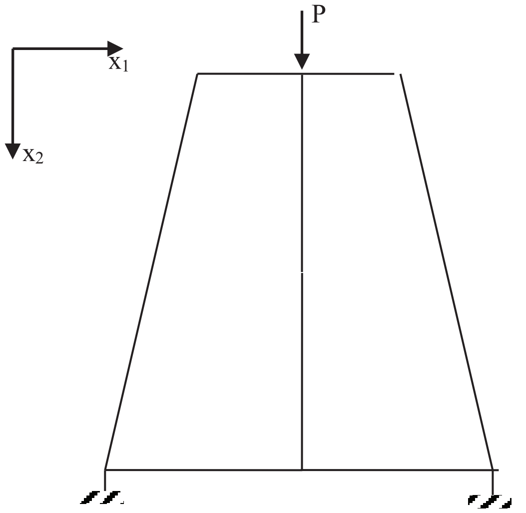


Figure 10. Design scheme.

Design scheme of a trapezoidal beams-wall is shown on Figure 10: height is equal to 140 cm; width at base is equal to 140 cm and at the top of the structure is equal to 70 cm; $E = 3 \cdot 10^3$ kN/cm^2 is the modulus of elasticity of the material of structure; $\nu = 0.16$ is Poisson's ratio of elasticity of the material of structure; $P = 100$ kN is parameter of concentrated load. Besides, we have fastening at the bottom corner points in both directions.

We can use finite element method for numerical solution of the considering problem. Thus, we can get resultant system of equations with respect to the nodal displacements (u_1 and u_2),

$$K\overline{U} = \overline{R}, \qquad (2.41)$$

where K is global stiffness matrix; $\overline{R}$ is global load vector; $\overline{U}$ is vector of unknown nodal displacements having the following structure,

$$\overline{U} = [\overline{u}_{1,1}^T \dots u_{1,N}^T \dots u_{N,1}^T .. u_{N,N}^T]^T; \qquad (2.42)$$

$$\overline{u}_{i,j} = \begin{bmatrix} u_1(x_{1,j}, x_{2,i}) \\ u_2(x_{1,j}, x_{2,i}) \end{bmatrix} \qquad (2.43)$$

are displacements u_1 and u_2 in the node (i, j), $i, j = 1, \dots, N$.

Let us denote P_{12} as the matrix operator of such a permutation, i.e.

$$\overline{V} = P_{12}\overline{U} \quad \text{or} \quad \overline{U} = P_{12}^T\overline{V}. \qquad (2.44)$$

As a result, the resolving system (13) is converted to the form

$$K_v\overline{V} = \overline{R}_v, \qquad (2.45)$$

where we have

$$K_v = P_{12}KP_{12}^T; \quad \overline{R}_v = P_{12}\overline{R}. \qquad (2.46)$$

Besides, we use the following block matrices in formulas (2.26)-(2.38):

$$\Phi_1^b = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_1 \end{bmatrix}; \qquad (2.47)$$

$$\Psi_{k,1}^b = \begin{bmatrix} \Psi_{k,1} & 0 \\ 0 & \Psi_{k,1} \end{bmatrix}, \quad k = 1, 2, 3. \qquad (2.48)$$

In accordance with the second equation of the formula (2.44), the solution of the initial problem (2.41) has the form:

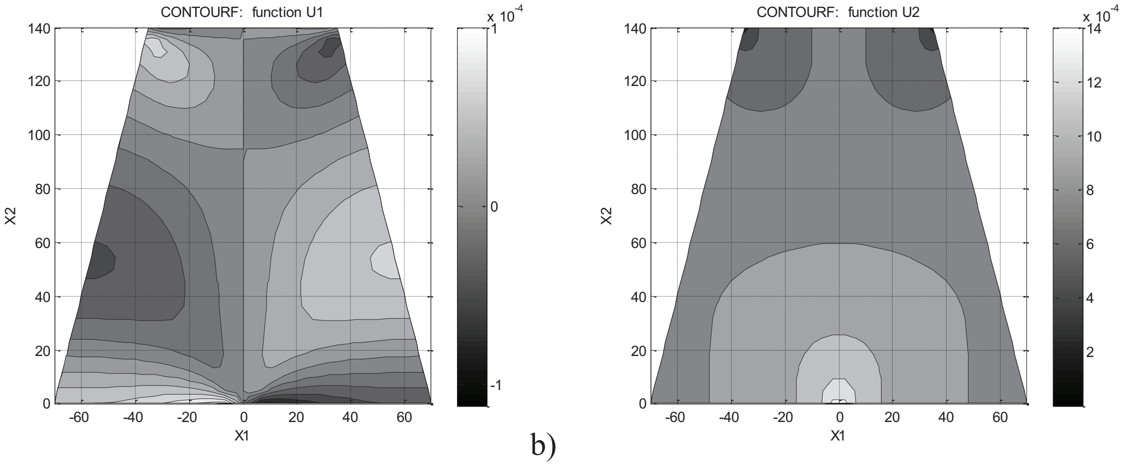
$$\overline{U} = P_{12}^T(\Phi_1^b\overline{w}_1 + \sum_{k=1}^{3} \Psi_{k1}^b\overline{w}_{k+1}). \qquad (2.49)$$

In accordance with the results of analysis graphs of displacements were obtained. Direct solution of the system (2.41) and solution defined by the formula (2.49) are presented at Figures 11-12.

3. THREE-DIMENSIONAL PROBLEMS

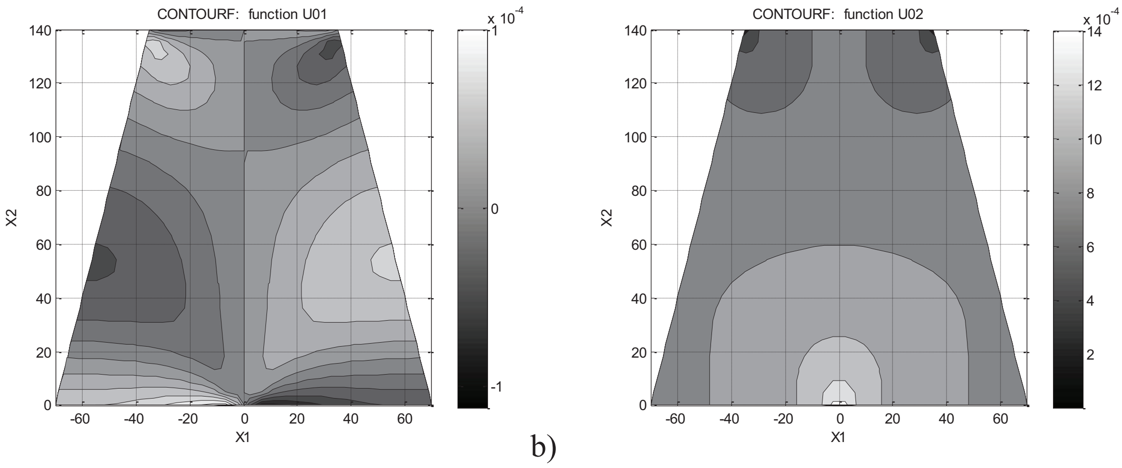
3.1. General information.

Let the initial three-dimensional domain be given as a rectangular prism [1]. Let L_1, L_2 and L_3 be lengths of sides of this prism in directions corresponding to Cartesian coordinates x_1, x_2 and x_3.



a) b)

Figure 11. Solution of the boundary problem (displacements u_1 (a) and u_2 (b)) in initial formulation.



a) b)

Figure 12. Solution of the boundary problem (displacements u_1 (a) and u_2 (b)) with the use of two-grid method.

We can use simple rectangular mesh (grid) for approximation of domain and divide each side of the rectangular prism (the initial domain) into $(N-1)$ equal parts. The corresponding mesh (grid) width are defined by formulas

$$h_1 = L_1/(N-1); \quad h_2 = L_2/(N-1); \quad h_3 = L_3/(N-1). \quad (3.1)$$

Thus, the resulting mesh contains N^3 nodes.
Let us introduce the mesh function

$$\bar{u} = [\, u_{1,1,1} \;\; \ldots \;\; u_{1,N,1} \;\; \ldots \;\; u_{N,N,1} \;\; \ldots \;\; u_{N,N,N} \,]^T . \quad (3.2)$$

We can represent the mesh function (1.2) in the form

$$\bar{u} = \sum_{j_3=1}^{N_0} \sum_{j_2=1}^{N_0} \sum_{j_1=1}^{N_0} u^0_{j_2,j_1,j_3} \overline{\Phi}^0_{j_2,j_1,j_3} , \quad (3.3)$$

where $\overline{\Phi}^0_{j_2,j_1,j_3}(i_2,i_1,i_3)$ is the (j_2,j_1,j_3)-th vector of a unit basis or a discrete zero-level Haar basis

$$\overline{\Phi}^0_{j_2,j_1,j_3}(i_2,i_1,i_3)= \\ =\begin{cases}1, & (i_1=j_1)\bigcap(i_2=j_2)\bigcap(i_3=j_3)\\ 0, & (i_1\neq j_1)\bigcup(i_2\neq j_2)\bigcup(i_3\neq j_3),\end{cases}$$

$$1\le j_2,j_1,j_3\le N_0=N;\quad 1\le i_2,i_1,i_3\le N_0=N; \tag{3.4}$$

$$u^0_{j_2,j_1,j_3}=u_{j_2,j_1,j_3},\quad 1\le j_1,j_2,j_3\le N; \tag{3.5}$$

$$N_0=N. \tag{3.6}$$

The mesh function (3.2) can also be represented in the form of an expansion in the Haar basis [1-7,9-25] of the first level:

$$\overline{u}=\sum_{j_3=1}^{N_1}\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1}u^1_{j_2,j_1,j_3}\overline{\Phi}^1_{j_2,j_1,j_3}+ \\ +\sum_{k=1}^{7}\sum_{j_3=1}^{N_1}\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1}v^1_{k,j_2,j_1,j_3}\overline{\Psi}^1_{k,j_2,j_1,j_3}, \tag{3.7}$$

where $\overline{\Phi}^1_{j_2,j_1,j_3}$ is (j_2,j_1,j_3)-th approximating vector of the discrete Haar basis of the first level,

$$\overline{\Phi}^1_{j_2,j_1,j_3}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3-1}+\overline{\Phi}^0_{2j_2-1,2j_1,2j_3-1}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1,2j_3-1}+\overline{\Phi}^0_{2j_2,2j_1,2j_3-1}+ \\ +\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3}+\overline{\Phi}^0_{2j_2-1,2j_1,2j_3}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1,2j_3}+\overline{\Phi}^0_{2j_2,2j_1,2j_3}); \tag{3.8}$$

$\overline{\Psi}^1_{k,j_2,j_1,j_3}$ – (k,j_2,j_1,j_3)-th refining vector of the discrete Haar basis of the first level,

$$\overline{\Psi}^1_{1,j_2,j_1,j_3}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3-1}-\overline{\Phi}^0_{2j_2-1,2j_1,2j_3-1}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1,2j_3-1}-\overline{\Phi}^0_{2j_2,2j_1,2j_3-1}+ \\ +\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3}-\overline{\Phi}^0_{2j_2-1,2j_1,2j_3}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1,2j_3}-\overline{\Phi}^0_{2j_2,2j_1,2j_3}); \tag{3.9}$$

$$\overline{\Psi}^1_{2,j_2,j_1,j_3}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3-1}+\overline{\Phi}^0_{2j_2-1,2j_1,2j_3-1}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1,2j_3-1}-\overline{\Phi}^0_{2j_2,2j_1,2j_3-1}+ \\ +\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3}+\overline{\Phi}^0_{2j_2-1,2j_1,2j_3}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1,2j_3}-\overline{\Phi}^0_{2j_2,2j_1,2j_3}); \tag{3.10}$$

$$\overline{\Psi}^1_{3,j_2,j_1,j_3}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3-1}-\overline{\Phi}^0_{2j_2-1,2j_1,2j_3-1}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1,2j_3-1}+\overline{\Phi}^0_{2j_2,2j_1,2j_3-1}+ \\ +\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3}-\overline{\Phi}^0_{2j_2-1,2j_1,2j_3}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1,2j_3}+\overline{\Phi}^0_{2j_2,2j_1,2j_3}); \tag{3.11}$$

$$\overline{\Psi}^1_{4,j_2,j_1,j_3}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3-1}+\overline{\Phi}^0_{2j_2-1,2j_1,2j_3-1}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1,2j_3-1}+\overline{\Phi}^0_{2j_2,2j_1,2j_3-1}- \\ -\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3}-\overline{\Phi}^0_{2j_2-1,2j_1,2j_3}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1,2j_3}-\overline{\Phi}^0_{2j_2,2j_1,2j_3}); \tag{3.12}$$

$$\overline{\Psi}^1_{5,j_2,j_1,j_3}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3-1}-\overline{\Phi}^0_{2j_2-1,2j_1,2j_3-1}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1,2j_3-1}-\overline{\Phi}^0_{2j_2,2j_1,2j_3-1}- \\ -\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3}+\overline{\Phi}^0_{2j_2-1,2j_1,2j_3}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1,2j_3}+\overline{\Phi}^0_{2j_2,2j_1,2j_3}); \tag{3.13}$$

$$\overline{\Psi}^1_{6,j_2,j_1,j_3}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3-1}+\overline{\Phi}^0_{2j_2-1,2j_1,2j_3-1}- \\ -\overline{\Phi}^0_{2j_2,2j_1-1,2j_3-1}-\overline{\Phi}^0_{2j_2,2j_1,2j_3-1}- \\ -\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3}-\overline{\Phi}^0_{2j_2-1,2j_1,2j_3}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1,2j_3}+\overline{\Phi}^0_{2j_2,2j_1,2j_3}); \tag{3.14}$$

$$\overline{\Psi}^1_{7,j_2,j_1,j_3}=\alpha(\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3-1}-\overline{\Phi}^0_{2j_2-1,2j_1,2j_3-1} \\ -\overline{\Phi}^0_{2j_2,2j_1-1,2j_3-1}+\overline{\Phi}^0_{2j_2,2j_1,2j_3-1}- \\ -\overline{\Phi}^0_{2j_2-1,2j_1-1,2j_3}+\overline{\Phi}^0_{2j_2-1,2j_1,2j_3}+ \\ +\overline{\Phi}^0_{2j_2,2j_1-1,2j_3}-\overline{\Phi}^0_{2j_2,2j_1,2j_3}); \tag{3.15}$$

$$u^1_{j_2,j_1,j_3}=(\overline{u},\overline{\Phi}^1_{j_2,j_1,j_3});$$

$$v^1_{k,j_2,j_1,j_3}=(\overline{u},\overline{\Psi}^1_{k,j_2,j_1,j_3}),\ k=1,...,7; \tag{3.16}$$

$$\alpha=1/(2\sqrt{2});\quad N_1=N/2 \tag{3.17}$$

α is normalizing factor.

In accordance with (3.7), the mesh function $\overline{u}$ is represented as a sum in which the first summand (the first sum) is its approximation on the grid of the first level including N_1^3 nodes, and the second term (the second sum) is called the refinement (the complement to the initial state) on the grid the first level. The representation (3.6) can be written in the form

$$\bar{u} = \bar{u}_1^0 + \sum_{k=1}^{7} \bar{v}_{k,1}^0 , \qquad (3.18)$$

$$\bar{u}_1^0 = \sum_{j_3=1}^{N_1}\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1} u_{j_2,j_1,j_3}^1 \bar{\Phi}_{j_2,j_1,j_3}^1 = \\ = \Phi_1(\Phi_1, \bar{u}) = \Phi_1\Phi_1^T\bar{u}; \qquad (3.19)$$

$$\bar{v}_{k,1}^0 = \sum_{j_3=1}^{N_1}\sum_{j_2=1}^{N_1}\sum_{j_1=1}^{N_1} v_{k,j_2,j_1,j_3}^1 \bar{\Psi}_{k,j_2,j_1,j_3}^1 = \\ = \Psi_{k,1}(\Psi_{k,1}, \bar{u}) = \Psi_{k,1}\Psi_{k,1}^T\bar{u}, \ k = 1,...,7; \qquad (3.20)$$

$$\Phi_1 = \\ = [\bar{\Phi}_{1,1,1}^1 \ \dots \ \bar{\Phi}_{1,N_1,1}^1 \ \dots \ \bar{\Phi}_{N_1,N_1,1}^1 \ \dots \ \bar{\Phi}_{N_1,N_1,N_1}^1]; \qquad (3.21)$$

$$\Psi_{k,1} = \\ = [\bar{\Psi}_{k,1,1,1}^1 \ \dots \ \bar{\Psi}_{k,1,N_1,1}^1 \ \dots \ \bar{\Psi}_{k,N_1,N_1,1}^1 \ \dots \ \bar{\Psi}_{k,N_1,N_1,N_1}^1], \\ k = 1,...,7. \qquad (3.22)$$

Due to the orthonormality of the Haar basis [2,3,5,6], the operators

$$P_\Phi = \Phi_1\Phi_1^T, \ \ P_{\Psi,k} = \Psi_{k,1}\Psi_{k,1}^T, \ \ k = 1,...,7 \qquad (3.23)$$

are projectors of the space of vector functions of the original grid to the space of their approximation on the first-level grid and its complement (the refining component) to the initial state, respectively.

3.2. Basic scheme of the two-grid method.
Let systems of linear algebraic equations

$$A\bar{u} = \bar{f} \qquad (3.24)$$

are discrete analogs of some operator equation defined on a given rectangular prism of order N^3.

We can substitute in (3.22) the expression for $\bar{u}$ in the form (3.16). Then we can multiply, in turn, both sides of the equality on the left by the matrices Φ_1^T and $\Psi_{k,1}^T$, $k = 1,2,3,4,5,6,7$. Thus we have

$$\Phi_1^T A\bar{u}_1^0 + \sum_{k=1}^{7} \Phi_1^T A\Psi_{k,1}\bar{v}_k^1 = \Phi_1^T \bar{f} , \qquad (3.25)$$

$$\Psi_{k,1}^T A\bar{u}_1^0 + \sum_{k=1}^{7} \Psi_{k,1}^T A\Psi_{k,1}\bar{v}_k^1 = \Psi_{k,1}^T \bar{f}, \ \ k = 1,...,7. \qquad (3.26)$$

Equations (3.25) and (3.26) can be combined in the system

$$\sum_{i=1}^{4} A_{i,j}\bar{w}_j = \bar{f}_i, \ \ i = 1,...,8. \qquad (3.27)$$

where $A_{i,j}$ are block matrices of size $N_1^3 \times N_1^3$; $\bar{f}_i$ and $\bar{w}_j$ are block vectors of size N_1^3,

$$A_{1,1} = \Phi_1^T A\Phi_1; \ \ A_{1,k+1} = \Phi_1^T A\Psi_{k,1}, \\ A_{k+1,1} = \Psi_{k,1}^T A\Phi_1, \ \ k = 1,...,7; \qquad (3.28)$$

$$A_{k+1,p+1} = \Psi_{k,1}^T A\Psi_{p+1}, \ \ k,p = 1,2,3,4,5,6,7; \qquad (3.29)$$

$$\bar{f}_1 = \Phi_1^T \bar{f}, \ \ \bar{f}_{k+1} = \Psi_{k,1}^T \bar{f}, \ \ k = 1,...,7; \qquad (3.30)$$

$$\bar{w}_1 = \bar{u}^1, \ \ \bar{w}_{k+1} = v_k^1, \ \ k = 1,...,7. \qquad (3.31)$$

We can find the solution of the system (3.27) using the block Gaussian method.
The expanded block matrix has the form:

$$\left[\begin{array}{cccccccc|c}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & A_{1,8} & \bar{f}_1 \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & A_{2,8} & \bar{f}_2 \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & A_{3,8} & \bar{f}_3 \\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} & A_{4,7} & A_{4,8} & \bar{f}_4 \\
A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & A_{5,5} & A_{5,6} & A_{5,7} & A_{5,8} & \bar{f}_5 \\
A_{6,1} & A_{6,2} & A_{6,3} & A_{6,4} & A_{6,5} & A_{6,6} & A_{6,7} & A_{6,8} & \bar{f}_6 \\
A_{7,1} & A_{7,2} & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} & A_{7,8} & \bar{f}_7 \\
A_{8,1} & A_{8,2} & A_{8,3} & A_{8,4} & A_{8,5} & A_{8,6} & A_{8,7} & A_{8,8} & \bar{f}_8
\end{array}\right]. \qquad (3.32)$$

We have the following result of forward algorithm:

$$\left[\begin{array}{cccccccc|c} A^0_{1,1} & A^0_{1,2} & A^0_{1,3} & A^0_{1,4} & A^0_{1,5} & A^0_{1,6} & A^0_{1,7} & A^0_{1,8} & \bar{f}^0_1 \\ 0 & A^1_{2,2} & A^1_{2,3} & A^1_{2,4} & A^1_{2,5} & A^1_{2,6} & A^1_{2,7} & A^1_{2,8} & \bar{f}^1_2 \\ 0 & 0 & A^2_{3,3} & A^2_{3,4} & A^2_{3,5} & A^2_{3,6} & A^2_{3,7} & A^2_{3,8} & \bar{f}^2_3 \\ 0 & 0 & 0 & A^3_{4,4} & A^3_{4,5} & A^3_{4,6} & A^3_{4,7} & A^3_{4,8} & \bar{f}^3_4 \\ 0 & 0 & 0 & 0 & A^4_{5,5} & A^4_{5,6} & A^4_{5,7} & A^4_{5,8} & \bar{f}^4_5 \\ 0 & 0 & 0 & 0 & 0 & A^5_{6,6} & A^5_{6,7} & A^5_{6,8} & \bar{f}^5_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & A^6_{7,7} & A^6_{7,8} & \bar{f}^6_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^7_{8,8} & \bar{f}^7_8 \end{array}\right], \tag{3.33}$$

where

$$A^k_{i,j} = A^{k-1}_{i,j} - C_{i,k} A^{k-1}_{k,j}, \quad k = 1,...,7, \quad i = k+1,...,8, \quad j = k+1,...,8; \tag{3.34}$$

$$\bar{f}^k_i = \bar{f}^{k-1}_i - C_{i,k} \bar{f}^{k-1}_k, \quad k = 1,...,7, \quad i = k+1,...,8; \tag{3.35}$$

$$C_{i,k} = A^{k-1}_{i,k} (A^{k-1}_{k,k})^{-1}, \quad k = 1,...,7, \quad i = k+1,...,8. \tag{3.36}$$

We have the following result of backward algorithm

$$\overline{w}_8 = (A^7_{8,8})^{-1} \bar{f}^7_8; \tag{3.37}$$

$$\overline{w}_i = (A^{i-1}_{i,i})^{-1} \left(\bar{f}^{i-1}_i - \sum_{j=i+1}^{4} A^{i-1}_{i,i} \overline{w}_j \right), \quad i = 7,...,1. \tag{3.38}$$

In accordance with formulas (3.18)-(3.20) solution of the considering problem (3.24) has the form

$$\overline{u} = \overline{u}^0_1 + \sum_{k=1}^{7} \overline{v}^0_{k,1} = = \Phi_1 \overline{u}^1 + \sum_{k=1}^{7} \Psi_{k,1} \overline{v}^1_k = \Phi_1 \overline{w}_1 + \sum_{k=1}^{7} \Psi_{k,1} \overline{w}_{k+1}. \tag{3.39}$$

It should be noted that the proposed method makes it possible to obtain a solution with the use of matrices of size $N_1^3 \times N_1^3$ and vectors of size N_1^3.

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