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ON THE USE OF EXTENDED PLATE THEORIES OF VEKUA – AMOSOV TYPE FOR WAVE DISPERSION PROBLEMS

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Abstract: The extended plate theory of I.N. Vekua – A.A. Amosov type is constructed on the background of the dimensional reduction approach and the Lagrangian variational formalism of analytical dynamics. The proposed theory allows one to obtain the hierarchy of refined plate models of different orders and to satisfy the boundary conditions on plates' faces exactly by introducing the corresponding constraint equations into the Lagrangian model of two-dimensional continuum. The normal wave dispersion in an elastic layer is considered, the convergence of the two-dimensional solutions to the exact one is studied for the locking phase frequencies, the dimensionless stress distributions across the thickness of a layer are shown.

Keywords: hierarchical modeling of plates, dimensional reduction, analytical continuum dynamics, constrained Lagrangian systems, elastic layer, wave dispersion

О ПРИМЕНЕНИИ РАСШИРЕННЫХ ТЕОРИЙ ПЛАСТИН И.Н. ВЕКУА – А.А. АМОСОВА К ЗАДАЧАМ О ДИСПЕРСИИ ВОЛН

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Аннотация: Расширенная теория пластин И.Н. Векуа – А.А. Амосова построена на базе метода пространственной редукции задачи механики деформируемого твердого тела и Лагранжева вариационного формализма аналитической динамики континуальных систем. Предложенная общая теория позволяет получить иерархическую систему моделей пластин различных порядков и притом точно удовлетворить краевым условиям на лицевых поверхностях, вводя соответствующие краевым условиям уравнения связей в Лагранжеву формулировку модели двумерного континуума. Рассмотрена задача о дисперсии нормальных волн в упругом слое, изучена сходимость последовательности решений двумерных задач к точному решению трехмерной задачи теории упругости по частотам запирания распространяющихся мод, приведены безразмерные распределения компонентов тензора напряжения, соответствующие собственным функциям, по толщине слоя.

Ключевые слова: иерархические модели пластин, редукция пространственной размерности, аналитическая динамика континуума, Лагранжевы системы со связями, слой упругий, нормальных волн дисперсия

INTRODUCTION

An accurate and reliable modeling of high-frequency and wave dynamics of various thin-walled structures requires refined shell and plate models [1] accounting for higher-order degrees of freedom and of transverse normal strains [2] besides the translation and rotation of mid-surface's points and the transverse shear. Indeed, essentially three-dimensional stress and strain states were found even in homogeneous isotropic plates [3, 4] and conical shells [5]; such a strain state with boundary layers appearing near the faces could be properly described by at least fourth-order polynomial approximation of displacement field [3] even for a thin plate. Thus, various traditional models of shear-deformable shells and plates fail in the case of high-frequency dynamics while the refined models based on special function expansions [4] or power series expansions [5] may be quite efficient. Moreover three-dimensional field distributions should be accounted for laminated structures [2, 6, 7] due to the effects of anisotropy and weak transverse stiffness. The detail survey of refined theories for laminated structures is presented in [8] by Carrera who proposed the “Carrera Unified Formulation” of refined shell models [7].

Another topical problem that needs for higher order shell and plate modeling consists in the modeling of dynamics of functionally graded structures with high heterogeneity across their thickness [9]. The known exact solutions obtained for such structures primarily in statics (e. g. see [10]) could be useful as appropriate reference solutions. It has to be noted that only a few exact solutions were obtained for free vibration problems for graded plates [11]. Thus, reliable approximate models are strictly required; the higher-order shear theories accounting for the transverse normal strain were used [12, 13], including non-polynomial based ones [14]. On the other hand, the proper choice of assumption-based plate model for reliable modeling of high frequency dynamics could become too complex. Thus, the power series expansion [5, 15, 16], generalized Fourier expansions [17], or sampling surfaces approach [18] resulting in full hierarches of two-dimensional models may be efficient alternatives in mechanics of heterogeneous thin-walled structures. Let us also note that the hierarchical approach could be very useful for wave dispersion problems where the use of different plate theories should be well-founded [19]. Such an approach based on the use of Legendre polynomials as expansion functions was developed in [20, 21]. This method results in the classical spectral problem statement; it has shown its reliability and simplicity and was further used for Lamb wave dispersion problem in graded plates [22], moreover its state-space formalism was developed recently [23]. Thus, the hierarchical modeling could be also efficient in such problems for heterogeneous thin-walled structures as boundary resonances [24] or edge waves analysis [25-27] where three-dimensional elasticity solutions or higher-order asymptotic approximations may become too complex.

On the other hand, the general dimensional reduction approach based on the generalized Fourier series results in the one of most general shell theories proposed by I.N. Vekua [28]. Its efficiency in various transient dynamics problems for shells was shown, for instance, in [29]. The further development of the hierarchical theory of shells of Vekua type was proposed by A.A. Amosov [30]; in particular the exact dependency of the metrics on the normal coordinate was taken into account. The Amosov’s formalism [31] is based on the Galerkin method and on the tensor algebra of linear operators in Hilbert spaces; such a formulation make it possible, among other things, to use powerful computer algebra systems to construct higher-order models of shells of variable thickness and curvature and allows one the automation level close to the finite element modeling. Several other versions of the Vekua’s approach were also proposed recently (e. g. see [32-34]).

The Amosov’s shell theory results in the set of various models of two-dimensional Lagrangian systems in terms of the analytical dynamics of continua. Indeed, the continuum system can be defined on a two-dimensional manifold within a set of field variables and the Lagrangian density; the type of a theory depends on the definition of the field variables. In particular, the use of expansion factors of the three-dimensional displacement field with respect to the orthogonal function system of normal coordinate as field variables leads to the classical higher-order shell theory [28-34]. As well Legendre polynomials as trigonometric expansion functions could be used under the same formalism; in other word the theory [31] secures the covariance of the governing equations with respect to the expansion system. The further development consists in the biorthogonal expansion of the displacement vector; it allows the introduction of finite element, or finite layer discretization similar to [18] into the unified formulation of the shell theory of A.A. Amosov's type [35].

The higher-order plate models obtained in terms of the variational formalism [35] were validated on the background of the well-known Rayleigh-Lamb problem of wave dispersion in a plane elastic layer [36]. The satisfying convergence of approximate solutions given by plate models to the exact one [36] was shown for phase frequencies [37, 39] and for normal waveforms [38-42].

It must be noted nevertheless that the so-called "elementary" theory [42] allows one to satisfy the boundary conditions on the faces of a plate only approximately as a result of the convergence of a sequence of two-dimensional solutions [31, 43], therefore the reflection condition on the face of a layer cannot be secured for all computed wave forms. This drawback can be eliminated in "extended" higher-order theories; the earliest one was based on the use of series residuals as supplementary variables [28]. The analytical dynamics formalism offer another possibility. Indeed, the boundary conditions shifted from the faces onto the base surface become constraint equations for the Lagrangian system [44, 45]. Thus, the dynamic equations can be derived by means of the Lagrange multipliers method [44, 45] that allows one to obtain asymptotically consistent low-order models [46, 47]. On the other hand, the spectral problem of the wave dispersion is can be solved using the approach [48]; the use of this method and the accuracy of extended higher-order plate models is shown below.

1. BASICS OF THE EXTENDED PLATE THEORY OF N-TH ORDER

Let us consider a plate as a three-dimensional body of thickness $2h$, bounded by faces $S_{\pm}$ and a lateral surface S_B, and let its two-dimensional model be defined on the manifold S, corresponding to the mid-plane and furnished by the curvilinear coordinates

$$\xi^{\alpha} \in D_{\xi} \subseteq \square^{2},\ \alpha = 1,2$$

[35, 46, 49]. A Lagrangian continuum system on S can be defined within the configuration manifold Ω_N, the set of field variables $u_{\alpha}^{(k)}$, $u_3^{(k)}$ being the biorthogonal expansion factors for the displacement $\mathbf{u}$ with respect to the system $p_{(k)}(\zeta)$, $p^{(k)}(\zeta)$ [35, 46],

$$(p_{(k)}, p^{(m)})_1 = \delta_{(k)}^{(m)};$$

$$\mathbf{u}\left(t,\xi^{\alpha},\zeta\right) \approx \left(u_{\alpha}^{(k)}\mathbf{r}^{\alpha} + u_3^{(k)}\mathbf{n}\right) p_{(k)}\left(\zeta\right), k = \overline{0,N};$$

$$u_{\alpha}^{(k)} = \left(u_{\alpha}, p^{(k)}\right)_1 \equiv \int_{-1}^{1} u_{\alpha}\left(t,\xi^{\alpha},\zeta\right) p^{(k)}\left(\zeta\right) d\zeta, \ldots$$

here $\zeta \in [-1,1]$ denotes the dimensionless normal coordinate. The surface Lagrangian densitiy can be written as follows [49]:

$$\begin{aligned}
&\mathcal{L}_{S}\left(u_{\alpha}^{(k)}, u_3^{(k)}, \dot{u}_{\alpha}^{(k)}, \dot{u}_3^{(k)}, \bar{\bar{\nabla}}_{\beta} u_{\alpha}^{(k)}, \bar{\bar{\nabla}}_{\beta} u_3^{(k)}\right) = \\
&= \tfrac{1}{2}\rho_{(k)}^{(m)}\left(\dot{u}_{(m)}^{\alpha}\dot{u}_{\alpha}^{(k)} + \dot{u}_{(m)}^{3}\dot{u}_3^{(k)}\right) + F_{(k)}^{i}u_i^{(k)} - \\
&- \tfrac{1}{2}\left(C_{(km)}^{\alpha\beta\gamma\delta}\bar{\bar{\nabla}}_{\gamma}u_{\delta}^{(m)} + C_{(km)}^{\alpha\beta 3}u_3^{(m)}\right)\bar{\bar{\nabla}}_{\beta}u_{\alpha}^{(k)} - \\
&- \tfrac{1}{2}\left(C_{(km)}^{3\beta\gamma 3}\bar{\bar{\nabla}}_{\gamma}u_3^{(m)} + C_{(km)}^{3\beta\gamma}u_{\gamma}^{(m)}\right)\bar{\bar{\nabla}}_{\beta}u_3^{(k)} - \\
&- \tfrac{1}{2}h^{-1}\left(C_{(km)}^{33\gamma\delta}\bar{\bar{\nabla}}_{\gamma}u_{\delta}^{(m)} + C_{(km)}^{333}u_3^{(m)}\right)\bar{D}_{(n\square)}^{(\square k)}u_3^{(n)} - \\
&- \tfrac{1}{2}h^{-1}\left(C_{(km)}^{\alpha 3\gamma 3}\bar{\bar{\nabla}}_{\gamma}u_3^{(m)} + C_{(km)}^{\alpha 3\gamma}u_{\gamma}^{(m)}\right)\bar{D}_{(n\square)}^{(\square k)}u_{\alpha}^{(n)};
\end{aligned} \tag{1.1}$$

while the contour density is represented as

$$\mathcal{L}_{\partial S}\left(u_{\alpha}^{(k)}, u_3^{(k)}\right) = q_{B(k)}^{\alpha}u_{\alpha}^{(k)} + q_{B(k)}^{\alpha}u_{\alpha}^{(k)}. \tag{1.2}$$

Here $\rho_{(k)}^{(m)}$, $D_{(n\square)}^{(\square k)}$, $C_{(km)}^{\alpha\beta\gamma\delta}$, $C_{(km)}^{33\gamma\delta}$, $C_{(km)}^{\alpha 3\gamma 3}$, $C_{(km)}^{\alpha\beta 3}$, $C_{(km)}^{3\beta\gamma}$, $C_{(km)}^{333}$,… are linear operators [31,35,43]:

$$\rho^{(m)}_{(k)} = \left(\rho p_{(k)}, p^{(m)}\right)_1; \quad D^{(\cdot k)}_{(n\cdot)} = \left(\tfrac{d}{d\zeta} p_{(n)}, p^{(k)}\right)_1;$$
$$C^{\alpha\beta 3}_{(km)} = h^{-1} D^{(\cdot n)}_{(k\cdot)} C^{\alpha\beta 33}_{(nm)}; \quad C^{3\beta\gamma}_{(km)} = h^{-1} D^{(\cdot n)}_{(k\cdot)} C^{3\beta 3\gamma}_{(nm)};$$
$$C^{333}_{(km)} = h^{-1} D^{(\cdot n)}_{(k\cdot)} C^{3333}_{(nm)};$$
$$C^{\alpha\beta\gamma\delta}_{(km)} = \left(C^{\alpha\beta\gamma\delta} p_{(k)}, p_{(m)}\right)_1, \ldots$$

The following constraint equations follow from the boundary conditions on $S_{\pm}$ [44-46]:

$$\begin{gathered} C^{i\alpha\beta}_{\pm(k)} \overline{\nabla}_\beta u^{(k)}_\alpha + C^{i\beta 3}_{\pm(k)} \overline{\nabla}_\beta u^{(k)}_3 + \\ + h^{-1} D^{(\cdot k)}_{(m\cdot)} \left(C^{i3\alpha}_{\pm(k)} u^{(m)}_\alpha + C^{i33}_{\pm(k)} u^{(m)}_3 \right) = q^i_{\pm}; \\ C^{i\beta j}_{\pm(k)} = C^{i3j\beta} \big|_{\zeta=\pm 1} \, p_{(k)}\left(\pm 1\right). \end{gathered} \tag{1.3}$$

Thus, the plate model is defined as a two-dimensional continuum within the field variables $u^{(k)}_\alpha$, $u^{(k)}_3$ the Lagrangian densities (1.1), (1.2) and the constraints (1.3). The appropriate equations of dynamics could be obtained as generalized Lagrange equations of the second kind [44, 45] by means of Lagrange multiplier method. They coinside with the "elementary" theory [35] if the constraints (1.3) are neglected:

$$\begin{aligned} \rho^{(m)}_{(k)} \ddot{u}^{\alpha}_{(m)} &= \overline{\nabla}_\beta \sigma^{\alpha\beta}_{(k)} - h^{-1} D^{(\cdot m)}_{(k\cdot)} \sigma^{\alpha 3}_{(m)} + F^{\alpha}_{(k)}; \\ \rho^{(m)}_{(k)} \ddot{u}^{3}_{(m)} &= \overline{\nabla}_\beta \sigma^{3\beta}_{(k)} - h^{-1} D^{(\cdot m)}_{(k\cdot)} \sigma^{33}_{(m)} + F^{3}_{(k)}; \end{aligned} \tag{1.4}$$

The problem statement is closed by the boundary conditions (1.5) on the contour ∂S

$$\begin{aligned} \left(\sigma^{\alpha\beta}_{(k)} \nu_\beta - q^{\alpha}_{B(k)}\right) \delta u^{(k)}_\alpha \Big|_{S_B} &= 0; \\ \left(\sigma^{3\beta}_{(k)} \nu_\beta - q^{3}_{B(k)}\right) \delta u^{(k)}_3 \Big|_{S_B} &= 0; \end{aligned} \tag{1.5}$$

and the initial conditions (1.6):

$$\begin{aligned} u^{(k)}_\alpha \Big|_{t=t_0} &= U^{(k)}_\alpha; \; u^{(k)}_3 \Big|_{t=t_0} = U^{(k)}_3; \\ \dot{u}^{(k)}_\alpha \Big|_{t=t_0} &= V^{(k)}_\alpha; \; \dot{u}^{(k)}_3 \Big|_{t=t_0} = V^{(k)}_3. \end{aligned} \tag{1.6}$$

2. STATEMENT OF THE WAVE DISPERSION PROBLEM FOR THE EXTENDED PLATE THEORY

Let us consider hence a plane isotropic layer:

$$\begin{gathered} C^{\alpha\beta\gamma\delta}_{(km)} = G_{(km)} \left\lfloor \lambda a^{\alpha\beta} a^{\gamma\delta} + \mu\left(a^{\alpha\gamma} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\gamma}\right) \right\rfloor; \\ C^{3333}_{(km)} = G_{(km)}\left(\lambda + 2\mu\right); \quad C^{\alpha 3\beta 3}_{(km)} = G_{(km)} \mu a^{\alpha\beta}; \\ C^{\alpha\beta 33}_{(km)} = G_{(km)} \lambda a^{\alpha\beta}; \quad G_{(km)} = \left(p_{(k)}, p_{(m)}\right)_1. \end{gathered}$$

Let the wave propagate along the axis Ox_1 of the Cartesian frame and let us introduce the following dimensionless variables:

$$\begin{gathered} \xi = x_1 h^{-1}; \quad \zeta = x_2 h^{-1}; \quad \tau = t c_2 h^{-1}; \\ \tilde{u}^{(k)}_{\alpha\beta} = u^{(k)}_\alpha h^{-1}; \quad \tilde{\sigma}^{(k)}_{\alpha\beta} = \left(\rho c_2^2\right)^{-1} \sigma^{(k)}_{\alpha\beta}, \\ c_2 = \mu \rho^{-1} \end{gathered} \tag{1.7}$$

is the shear wave velocity. As a result, we obtain the following dimensionless dynamic equations for the Nth order theory:

$$\begin{gathered} \partial^2_\tau u^{(k)}_1 = \beta^{-2} \partial^2_\xi u^{(k)}_1 + D^{(\cdot k)}_{(n\cdot)} \overline{D}^{(\cdot n)}_{(m\cdot)} u^{(m)}_1 - \\ - \left[D^{(k\cdot)}_{(\cdot m)} - \left(\beta^{-2} - 2\right) \overline{D}^{(\cdot k)}_{(m\cdot)} \right] u^{(m)}_2; \\ \partial^2_\tau u^{(k)}_2 = \partial^2_\xi u^{(k)}_2 + \beta^{-2} D^{(\cdot k)}_{(n\cdot)} \overline{D}^{(\cdot n)}_{(m\cdot)} u^{(m)}_2 - \\ - \left[\left(\beta^{-2} - 2\right) D^{(\cdot k)}_{(m\cdot)} - \overline{D}^{(\cdot k)}_{(m\cdot)} \right] u^{(m)}_2, \\ \beta^2 = \left(c_2 / c_1\right)^2 = \left(1 - 2\nu\right) / \left(2 - 2\nu\right), \\ c_1 = \left(\lambda + 2\mu\right) \rho^{-1} \end{gathered} \tag{1.8}$$

is the bulk wave velocity. The dimensionless constraint equations corresponding to (1.3) can be written as

$$\begin{gathered} \left\lfloor \left(\beta^{-2} - 2\right) \partial_\xi u^{(k)}_1 + \beta^{-2} \overline{D}^{(\cdot k)}_{(n\cdot)} u^{(n)}_3 \right\rfloor p^{\pm}_{(k)} = 0; \\ \left(\overline{D}^{(\cdot k)}_{(n\cdot)} u^{(n)}_1 + \partial_1 u^{(k)}_3 \right) p^{\pm}_{(k)} = 0 \\ p^{\pm}_{(k)} = p_{(k)}(\pm 1), \end{gathered} \tag{1.9}$$

the sign “+” corresponds to the upper face and the “–“ to the lower one; therefore we have two pairs of constraint equations on both faces.
Let us consider normal waves, i. e.

$$u_\alpha^{(k)} = U_\alpha^{(k)} \exp\left[i\left(\kappa\xi - \omega\tau\right)\right], \qquad (1.10)$$
$$\kappa = kh$$

is the dimensionless wavenumber and

$$\omega = \Omega h / c_2$$

is the dimensionless phase frequency. Substituting (1.10) into (1.8) we obtain the spectral problem (1.11) for the linear operator $\mathbf{A}$ given by (1.12) (see [37-42]):

$$\left(\mathbf{A} - \omega^2 \mathbf{I}\right) \cdot \mathbf{U} = 0, \quad \mathbf{U} = \left(U_1^{(m)} \; U_2^{(m)}\right), \qquad (1.11)$$

$$\mathbf{A}_{(2N+2)\times(2N+2)} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix};$$
$$\left(A_{(m)}^{(k)}\right)_{11} = 4\left(1-\beta^2\right)\kappa^2 \delta_{(m)}^{(k)} + D_{(\cdot n)}^{(k\cdot)} \bar{D}_{(\cdot m)}^{(n\cdot)};$$
$$\left(A_{(m)}^{(k)}\right)_{12} = i\kappa\left[D_{(\cdot m)}^{(k\cdot)} - 2\left(1-2\beta^2\right)\bar{D}_{(\cdot m)}^{(k\cdot)}\right]; \quad (1.12)$$
$$\left(A_{(m)}^{(k)}\right)_{21} = i\kappa\left[2\left(1-2\beta^2\right)D_{(\cdot m)}^{(k\cdot)} - \bar{D}_{(\cdot m)}^{(k\cdot)}\right];$$
$$\left(A_{(m)}^{(k)}\right)_{21} = \kappa^2 \delta_{(m)}^{(k)} + 4\left(1-\beta^2\right) D_{(\cdot n)}^{(k\cdot)} \bar{D}_{(\cdot m)}^{(n\cdot)}.$$

The linear constraints (1.9) can be hence formulated for the images $U_i^{(k)}$ as follows:

$$\mathbf{B} \cdot \mathbf{U} = 0, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_S \\ \mathbf{B}_A \end{pmatrix}_{4\times(2N+2)}; \qquad (1.13)$$
$$\mathbf{B}_S = \tfrac{1}{2}\left(\mathbf{B}_+ + \mathbf{B}_-\right), \quad \mathbf{B}_A = \tfrac{1}{2}\left(\mathbf{B}_+ - \mathbf{B}_-\right);$$
$$\mathbf{B}_\pm = \begin{pmatrix} i\kappa\left(\beta^{-2} - 2\right)\delta_{(n)}^{(k)} p_{(k)}^\pm & \beta^{-2}\bar{D}_{(n\square)}^{(\square k)} p_{(k)}^\pm \\ \bar{D}_{(n\square)}^{(\square k)} p_{(k)}^\pm & i\kappa\delta_{(n)}^{(k)} p_{(k)}^\pm \end{pmatrix}. \qquad (1.14)$$

Here the symmetry conditions for the constraints are taken into account. Considering hence for (1.13), (1.14), we obtain the solution of the constrained stationary values problem for two quadratic forms $\mathbf{A}$ and $\mathbf{I}$ accordingly to [48]:

$$\mathbf{U}^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{U} / \mathbf{U} \cdot \mathbf{U} = 0, \quad \mathbf{B} \cdot \mathbf{U} = 0. \qquad (1.15)$$

Following [48], let us introduce the QZ decomposition for the constraint matrix (1.13)

$$\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{B}^{\mathrm{T}} \cdot \mathbf{Z} = \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}_{(2N+2)\times 4}, \qquad (1.16)$$

As a result, we obtain the following operator accounting for the constraints (1.13):

$$\mathbf{A}_C = \mathbf{Q}^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{Q}, \qquad (1.17)$$
$$\mathbf{A}_C = \begin{pmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{pmatrix}_{(2N+2)\times(2N+2)}$$

and the stationary values for (1.15) can be found from the unconstrained problem:

$$\left(\bar{\mathbf{A}}_{22} - \omega^2 \mathbf{I}\right) \cdot \mathbf{V} = 0, \quad \mathbf{V} = \mathbf{Q}^{\mathrm{T}} \cdot \mathbf{U}. \qquad (1.18)$$

The spectrum of the system consists in two subspectra S and A corresponding to the longitudinal and bending waves in the layer:

$$S: k, m = \{2n, N+2n\};$$
$$A: k, m = \{2n-1, N+2n-2\},$$
$$n \in [0, [\tfrac{1}{2}(N-1)]] \cup \mathrm{Z}.$$

The waveforms can be defined hence in terms of the Nth order plate theory as follows [37]:

$$u_\alpha^n(\zeta) = U_\alpha^{kn} p_{(k)}(\zeta),$$
$$\alpha = 1, 2, \quad k = 0 \ldots N-1, \quad n = 1 \ldots N; \qquad (1.19)$$
$$\mathbf{U}^m = \mathbf{Q} \cdot \left\lfloor V_1^{km} \quad V_2^{km} \right\rfloor, \quad m \in \left[1, 2N-2\right] \cap \mathrm{Z}.$$

3. CONVERGENCY ANALYSIS

Accordingly to [37-42], let us use Legendre polynomials as expansion functions $p_{(k)}(\zeta)$. The locking phase frequencies computed as stationary values of the problem (1.15), or the eigenvalues for the spectral problem (1.18) at $\kappa \to 0$, are presented below in the Table 1 and Table 2 for longitudinal and bending waves respectively.

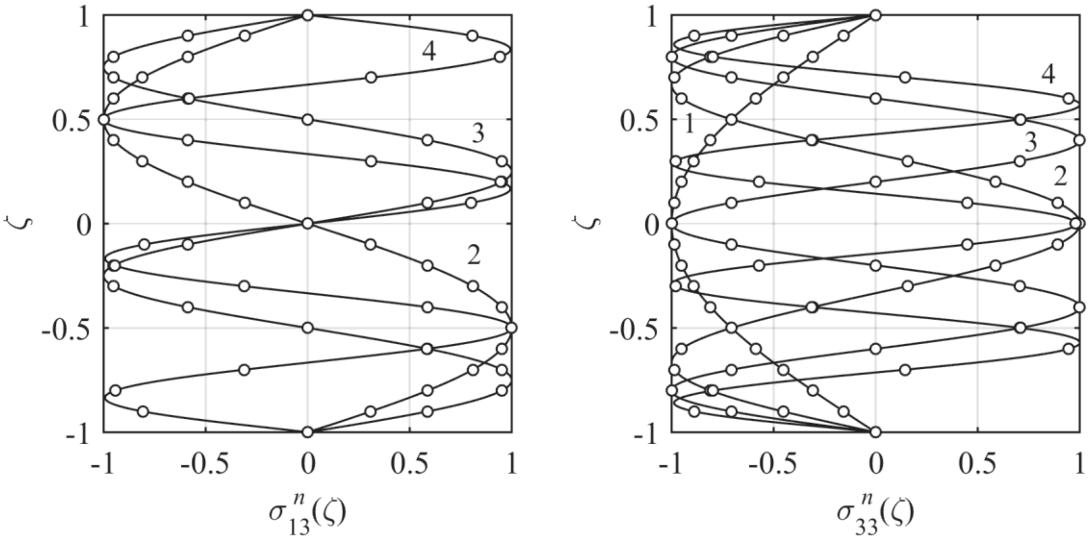
Table 1. Convergence of dimensionless locking frequencies of longitudinal waves to the Rayleigh-Lamb problem solution.

N \ n	1	2	3	4	5	6	7	8	9	10
2	0.00	–	–	–	–	–	–	–	–	–
3	0.00	1.89	–	–	–	–	–	–	–	–
4	0.00	1.89	1.97	–	–	–	–	–	–	–
5	0.00	1.91	1.98	5.68	–	–	–	–	–	–
6	0.00	1.91	2.00	4.03	5.68	–	–	–	–	–
7	0.00	1.91	2.00	4.03	5.74	9.83	–	–	–	–
8	0.00	1.91	2.00	4.02	5.73	6.37	9.82	–	–	–
9	0.00	1.91	2.00	4.02	5.72	6.37	9.64	14.67	–	–
10	0.00	1.91	2.00	4.00	5.72	6.11	9.13	9.64	14.67	–
11	0.00	1.91	2.00	4.00	5.72	6.11	9.13	9.51	13.73	20.40
12	0.00	1.91	2.00	4.00	5.72	6.00	8.35	9.52	12.40	13.73
13	0.00	1.91	2.00	4.00	5.72	6.00	8.35	9.53	12.40	13.32
14	0.00	1.91	2.00	4.00	5.72	6.00	8.00	9.53	10.84	13.32
15	0.00	1.91	2.00	4.00	5.72	6.00	8.00	9.53	10.84	13.34
16	0.00	1.91	2.00	4.00	5.72	6.00	8.00	9.53	10.05	13.34
Ex.	0.00	1.91	2.00	4.00	5.72	6.00	8.00	9.53	10.00	12.00

Table 2. Convergence of dimensionless locking frequencies of bending waves to the Rayleigh-Lamb problem solution.

N \ n	1	2	3	4	5	6	7	8	9	10
2	0.00	–	–	–	–	–	–	–	–	–
3	0.00	0.99	–	–	–	–	–	–	–	–
4	0.00	0.99	3.77	–	–	–	–	–	–	–
5	0.00	1.00	2.98	3.77	–	–	–	–	–	–
6	0.00	1.00	2.98	3.82	7.68	–	–	–	–	–
7	0.00	1.00	3.00	3.81	5.16	7.68	–	–	–	–
8	0.00	1.00	3.00	3.81	5.16	7.67	12.14	–	–	–
9	0.00	1.00	3.00	3.81	5.06	7.67	7.70	12.14	–	–
10	0.00	1.00	3.00	3.81	5.06	7.62	7.70	11.65	17.41	–
11	0.00	1.00	3.00	3.81	5.00	7.20	7.62	10.70	11.65	17.41
12	0.00	1.00	3.00	3.81	5.00	7.20	7.62	10.70	11.41	15.92
13	0.00	1.00	3.00	3.81	5.00	7.00	7.62	9.56	11.41	14.21
14	0.00	1.00	3.00	3.81	5.00	7.00	7.62	9.56	11.42	14.21
15	0.00	1.00	3.00	3.81	5.00	7.00	7.62	9.00	11.42	12.19

N \ n	1	2	3	4	5	6	7	8	9	10
16	0.00	1.00	3.00	3.81	5.00	7.00	7.62	9.00	11.41	12.19
Ex.	0.00	1.00	3.00	3.81	5.00	7.00	7.62	9.00	11.00	11.43



(a) (b)

Figure 1. Normal waveforms of longitudinal waves: exact solution [36] (solid line), plate theory of 15th order (dots).

The approximate solutions given by the extended plate theories converge to the exact solutions of the three-dimensional Rayleigh-Lamb problem [36] at the orders shown in the Table 3 (the convergence is estimated by the relative error computed at $\kappa \to 0$ as follows:

$$(\omega_0 - \omega_{0L})/\omega_{0L} \le 0,05 .$$

Table 3. The approximate solutions by the extended plate theories at various orders.

n	2	3	4	5	6	7	8	9
Longitudinal modes								
N	3	3	4	6	6	10	10	16
Bending modes								
N	3	5	5	9	11	9	15	12

Let us compute the stress profiles corresponding to the approximate wave forms obtained from the plate theory of Nth order:

$$\sigma^n_{13}(\zeta) = \left\lfloor \bar{D}^{(\cdot k)}_{(m\cdot)} U_1^{mn} + i\kappa U_3^{kn} \right\rfloor p_{(k)}(\zeta);$$

$$\sigma^n_{33}(\zeta) = \left[i\kappa\left(\beta^{-2}-2\right) U_1^{kn} + \beta^{-2} \bar{D}^{(\cdot k)}_{(m\cdot)} U_3^{mn} \right] p_{(k)}(\zeta).$$

The normalized stress forms corresponding to longitudinal waves are shown on the Figure 1.
It can be seen that for all stress profiles the homogeneous boundary condition on the faces of the layer are satisfied exactly.

CONCLUSIONS

The solution of the wave dispersion problem for the plane elastic layer is obtained on the background of the extended plate theory of Nth order. This theory allows the exact satisfaction of the boundary conditions on the faces of the layer, thus, the wave reflection condition on the faces is secured. The convergence of the approximate locking phase frequencies to the exact ones following from the solution of the three-dimensional Rayleigh-Lamb problem is shown.

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